Electrostatic Field Approximations and Implications for MEMS Devices

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Abstract

The study of mathematical models of electrostatically actuated microelectromechanical systems (MEMS) has relied upon simplified electrostatic field approximations to facilitate the analysis. These approximations are typically obtained by utilizing the small aspect ratio of such devices to simplify Laplace's equation. With the appropriate scaling, terms small in this aspect ratio are ignored. Such an approximation is not typically uniformly valid in the spatial variables. Here, this approximation is revisited and a uniformly valid asymptotic theory for a drum shaped electrostatically actuated MEMS device is presented. This asymptotic theory is incorporated into the mechanical part of the model and the effect of retaining typically ignored terms is discussed.

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1 Introduction

The advent of microelectromechanical systems (MEMS) has revolutionized numerous branches of science and industry. Already firmly established as an essential component of modern sensors, such as those used for automobile air-bag deployment, [1], MEMS are making inroads into areas as diverse as the biomedical industry, [2], space exploration, [3] and telecommunications, [4]. The science of electrostatics is of paramount importance to this technology. The use of electrostatic forces to provide locomotion for MEMS devices is employed in devices such as accelerometers, [1], optical switches, [5], microgrippers, [6], micro force gauges, [7], transducers, [8], and micro pumps, [9]. Since at the micro-level, the voltages required to produce significant forces are small, it is expected that electrostatics will play a continually increasing role in MEMS. Unfortunately, electrostatic actuation suffers from an instability, limiting its effectiveness. This was known to even the earliest MEMS researchers such as Nathanson et. al., [10], who in 1967 first identified and described the ubiquitous "pull-in" voltage instability. In this instability, when applied voltages are increased beyond a certain critical voltage there is no longer a steady-state configuration of the device where mechanical members remain separate. This instability severely restricts the range of stable operation of many devices, [1, 5, 6, 7, 8, 9]. Consequently, the understanding and control of this instability presents a challenge of great technological importance.

This understanding lies in the intimate connection between electrostatic and elastic forces present in such devices, hence it is worthwhile to closely examine typical approximations made by researchers in this area. The major approximation typically used by MEMS researchers is what we shall refer to as the "local parallel plate approximation." Essentially, the fact the most MEMS devices possess a small aspect ratio is exploited. The small aspect ratio suggests that the electrostatic field may be locally approximated as if the field were due to two infinite parallel plates. Mathematically, one ignores terms small in the aspect ratio. Unfortunately, this approach is not completely justified from the mathematical point of view. In fact, in general there are regions of the domain for which correction terms to the approximation arise which are as big as the approximation itself. That is, the problem is a singular perturbation problem. Here, we develop an approximate electrostatic field solution for a drum shaped MEMS device which is uniformly valid throughout the domain. We use this uniform approximation to com-
pute the force on our device. We show that the force computed from this uniform approximation includes three types of terms. First, the typically retained term is present. Second, there are terms, due to fringing fields, which contribute near the boundaries of the domain. Finally, there is a term that corrects for the fact that electric fields become intense near corners. On the basis of these observations we propose a “corner corrected local parallel plate approximation.” We compare solutions to the elastic problem for the drum which rely upon the local parallel plate approximation with those obtained from the new approximate theory. We show that including the corner correction terms may have a dramatic effect on the structure of the bifurcation diagram for the elastic problem and hence on MEMS device behavior.

2 Formulation of the Model

In this section we present the governing equations for the behavior of our idealized drum shaped electrostatically actuated MEMS device. Our device consists of a thin circular elastic membrane suspended above a rigid plate. Both the membrane and the plate have radius $a$ and are separated by a gap of height $L$. The sidewalls supporting the membrane are electrically connected to the plate but are separated from the membrane by a thin insulating layer. In this way, a potential difference is applied between the membrane and the rest of the device. This ”drum” shaped geometry is sketched in Figure 1. With these assumptions in mind, we formulate the equations governing the electrostatic field. The electrostatic potential, $\phi$, satisfies

$$\nabla^2 \phi = 0$$ (1)

$$\phi(a, z') = 0 \quad z' \in [0, L]$$ (2)

$$\phi(r', 0) = 0 \quad r' \in [0, a]$$ (3)

$$\phi(r', u'(r')) = V \quad r' \in [0, a]$$ (4)

where here $u'(r')$ is the displacement of the membrane from $z' = L$. Note that we are assuming cylindrical symmetry, i.e. $u'$ is a function of $r'$ only.
Similarly, $\phi$ shall be assumed independent of $\theta$. The displacement of the membrane is assumed to satisfy

$$T \frac{d^2 u'}{dr'^2} + \frac{T}{r'} \frac{du'}{dr'} = \frac{\epsilon_0}{2} |\nabla \phi|^2 \tag{5}$$

Here, $T$ is the tension and $\epsilon_0$ the permittivity of free space. We assume the membrane is held fixed along its circumference and impose

$$u'(a) = L. \tag{6}$$

Additionally we impose

$$\frac{du'}{dr'}(0) = 0. \tag{7}$$
Next, we introduce dimensionless variables and rewrite our governing equations in dimensionless form. We define
\[ \psi = \phi/V, \quad u = u'/L, \quad r = r'/a, \quad z = z'/L \] (8)
and substitute these into equations (1)-(7). This yields
\[ \epsilon^2 \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = 0, \] (9)
\[ \psi(r, 0) = 0 \quad r \in [0, 1] \] (10)
\[ \psi(1, z) = 0 \quad z \in [0, 1] \] (11)
\[ \psi(r, u) = 1 \quad r \in [0, 1] \] (12)
\[ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = \lambda |\nabla \psi|^2 \] (13)
\[ u(1) = 1 \] (14)
\[ \frac{du}{dr}(0) = 0. \] (15)
Here \( \epsilon = L/a \) is an aspect ratio comparing device radius to gap size and \( \lambda = \epsilon_0 V_s^2 L^2 / 2 T \) is a dimensionless number which characterizes the relative strengths of electrostatic and mechanical forces in the system.

3 A Uniformly Valid Asymptotic Theory

In this section we focus on the electrostatic problem, equations (9)-(12), and derive a uniformly valid asymptotic approximation to the potential. We begin by making the small aspect ratio assumption, that is, we assume \( \epsilon \ll 1 \). If we simply send \( \epsilon \to 0 \) the partial differential equation in equation (9) reduces to
\[ \frac{\partial^2 \psi}{\partial z^2} = 0 \] (16)
which is easily integrated to yield
\[ \psi = \frac{z}{u}. \]  \hspace{1cm} (17)

Note that this is the approximate electrostatic field used by most author’s investigating electrostatically actuated MEMS, [6, 7, 9, 10, 11, 12, 13]. However, this solution is not valid near \( r = 1 \) as we cannot satisfy the condition that \( \psi(1, z) = 0 \). Hence, we take this solution as an outer solution in a boundary layer theory and attempt to insert a boundary layer at the walls by introducing the stretched variable
\[ \eta = \frac{1 - r}{\epsilon} \]

into equations (9)-(12). With this change of variables the problem in our inner region becomes
\[ \frac{\partial^2 \psi}{\partial \eta^2} - \frac{\epsilon}{1 - \epsilon \eta} \frac{\partial \psi}{\partial \eta} + \frac{\partial^2 \psi}{\partial z^2} = 0 \]  \hspace{1cm} (18)

\[ \psi(0, z) = 0 \]  \hspace{1cm} (19)

\[ \psi(\eta, 0) = 0 \]  \hspace{1cm} (20)

\[ \psi(\eta, u(1 - \epsilon \eta)) = 1. \]  \hspace{1cm} (21)

If we send \( \epsilon \to 0 \) here, we obtain
\[ \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \]  \hspace{1cm} (22)

\[ \psi(0, z) = 0 \]  \hspace{1cm} (23)

\[ \psi(\eta, 0) = 0 \]  \hspace{1cm} (24)

\[ \psi(\eta, 1) = 1. \]  \hspace{1cm} (25)
We need one more condition to complete the specification of the inner problem. This comes from matching to our outer solution. We impose
\[
\lim_{\eta \to \infty} \psi(\eta, z) = \lim_{r \to 1} \frac{z}{u(r)} = z. \tag{26}
\]
The inner or boundary layer problem, equation (22)-(26) is now easily solved. We find
\[
\psi(\eta, z) = z + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n\eta} \sin(n\pi z). \tag{27}
\]
Finally, adding our outer approximation, equation (17), to our inner approximation, equation (27), and subtracting off the common part, we obtain the following leading order uniformly valid asymptotic approximation to the electrostatic field
\[
\psi_{\text{uniform}}(\eta, z) = \frac{z}{u(r)} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n\pi \left(1 - \frac{z}{r}\right)} \sin(n\pi z). \tag{28}
\]

3.1 The Electrostatic Force

In order to compute the force on the membrane, we need to compute the quantity
\[
|\nabla \psi|^2 \tag{29}
\]
evaluated at \(z = u(r)\) as appears in equation (13). That is, we must compute
\[
|\nabla \psi|^2 = e^2 \left(\frac{\partial \psi}{\partial r}\right)^2_{z=u} + \left(\frac{\partial \psi}{\partial \zeta}\right)^2_{z=u}. \tag{30}
\]
Using our uniformly valid approximation, (28), to evaluate the force, we find
\[
|\nabla \psi|^2 = \frac{1}{u(r)^2} + e^2 \frac{u'(r)^2}{u(r)^2} \tag{31}
\]
\[
+ 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{u(r)} e^{-n\pi \left(1 - \frac{z}{r}\right)} \cos(n\pi u(r) - \epsilon u'(r) \sin(n\pi u(r)))
\]
\[
+ (2 \sum_{n=1}^{\infty} (-1)^n e^{-n\pi \left(1 - \frac{z}{r}\right)} \sin(n\pi u(r)))^2
\]
\[
+ (2 \sum_{n=1}^{\infty} (-1)^n e^{-n\pi \left(1 - \frac{z}{r}\right)} \cos(n\pi u(r)))^2.
\]
3.2 Approximate Elastic Theories

In order to complete the study of our electrostatically actuated MEMS device, the force computed in the previous subsection, equation (31), must be inserted into the elastic problem, equations (13)-(15) and the resulting problem for the displacement, \( u(r) \), analyzed. At this point it is worth considering commonly used approximations at the elastic level and examining equation (31) in order to evaluate the validity of these approximations. The standard approach, [6, 7, 9, 10, 11, 12], is to ignore all terms proportional to \( \epsilon \) in the force, equation (31), as well as fringing fields or edge effects. This implies that only the first term in equation (31) is retained. The resulting problem for the elastic displacement then simplifies to

\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = \frac{\lambda}{u(r)^2} \tag{32}
\]

\[
u(1) = 1 \tag{33}
\]

\[
\frac{du}{dr}(0) = 0. \tag{34}
\]

This approximation is what we refer to as the "local parallel plate approximation." Physically, this approximation assumes that at each point within the device, \((r, z)\), the electrostatic field is well approximated by assuming that it is due to a pair of parallel plates separated by a distance \( u(r) \). Examining equation (31), we see that this approximation is valid away from the \( O(\epsilon) \) boundary layer and provided that \( u'(r) \) remains \( O(1) \). The system, equations (32)-(34) was analyzed in [11, 12].

Further examination of the expression for force, equation (31), suggests additional approximations. The physical interpretation of the first term was outlined above. The second term in the expression may also be interpreted physically. As it is proportional to the gradient squared of \( u(r) \) it represents increased force near corners, i.e. in regions where the electric field becomes intense. Incorporating only the first and second terms of the force into the elastic problem yields

\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = \frac{\lambda (1 + \epsilon^2 u'^2)}{u(r)^2} \tag{35}
\]
\[ u(1) = 1 \]  \hspace{1cm} (36)

\[ \frac{du}{dr}(0) = 0. \]  \hspace{1cm} (37)

We refer to this approximation as the "corner corrected parallel plate approximation." As corners are likely to develop far away from the edges of the membrane we expect this new term to become most important in regions where the elastic forces are weakest. Hence, an effect on the structure of solutions to the elastic problem in likely. A comparison between the corner corrected theory and the local parallel plate theory is outlined in the next section.

Finally, the infinite sums appearing in the expression for force, equation (31) may be interpreted as representing edge effects or fringing fields. Note that these terms are exponentially small outside of a layer of thickness \( \epsilon \) located near \( r = 1 \). While one may propose a third approximate elastic theory which incorporates these terms, the fact that the membrane is held fixed at \( r = 1 \) suggests that the fringing fields will have little effect on the mechanical behavior of the membrane. That is, the fringing fields are important precisely in the region where the mechanical behavior of the membrane cannot be effected. On the other hand, incorporating all terms from equation (31) into an elastic theory would provide a useful test system against which to compare the results of various MEMS simulation tools.

## 4 Implications for Device Behavior

In this section we present preliminary numerical results comparing the local parallel plate and the corner corrected theories. As noted above, equations (32)-(34) were analyzed in [11, 12]. To perform the analysis, the symmetry in the equation was exploited. The principle result of the analysis was the construction of the bifurcation diagram shown in Figures 2,3. In typical bifurcation diagrams for electrostatically actuated MEMS, at most two solutions for a given applied voltage are possible. The surprising result of the analysis in [11, 12] was that for the drum model, equations (32)-(34), there were values of the applied voltage for which any number of steady-state solutions were possible.

The corner corrected theory, equations (35-37), lacks the symmetry present in equations (32)-(34) when \( \epsilon > 0 \). It is interesting to inquire as to how the
destruction of the symmetry effects the bifurcation diagram, Figures 2, 3. An easy way to visualize the effect of choosing $\epsilon > 0$ is to plot the meander for the boundary value problem. In this approach, which is simply a convenient way to visualize numerical shooting, the result of integrating the ode from $x = 0$ to $x = 1$, choosing all possible values of $u(0)$ is plotted. That is, the curve, $(u(1), u'(1))$, parameterized by $u(0)$ is plotted. Each time this curve intersects the line $u(1) = 1$, we have a solution to the boundary value problem. In Figure 4, for the $\epsilon = 0$ case, we see that this curve is a spiral. As $\lambda$ is varied, the spiral moves to the right producing more and more intersections with the line $u(1) = 0$ and hence resulting in more solutions to the boundary value problem. That is, the bifurcation diagram in Figures 2, 3 is produced. In Figures 5, 6 we plot the meander for choices of $\epsilon > 0$. We see that the pure spiral structure of the $\epsilon = 0$ case is destroyed. While the arm of the partial spiral still varies as a function of $\lambda$, only a finite number of solutions may be produced. Hence, we see that as a function of $\epsilon$ the multiplicity of the solutions changes. This is a numerical result, we conjecture that this is true in general, but a proof awaits further analysis.
5 Conclusions

We have examined the standard approximations used in analytical studies of electrostatically actuated MEMS devices. Recognizing that the electrostatic field approximation commonly used was not uniformly valid in the spatial variables, we developed a uniform approximation for a drum shaped geometry. Using this new approximation to compute the force on device components revealed that the force contained three physically distinct terms. Based on this observation a corrected theory was suggested. A preliminary numerical investigation of the corrected theory has shown that including previously neglected terms may dramatically effect the structure of solutions to the elastic problem. As a consequence device behavior is expected to depend on the aspect ratio of the system.

6 Acknowledgements

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Figure 4: Meander for the $\epsilon = 0$ case. Note the spiral continues, but is too tight to see in the figure. As $\lambda$ is increased the spiral arm moves to the right.

References


Figure 5: Meander for the $\epsilon = 0.4$ case. Note the spiral unraveled, limiting the number of possible solutions.


Figure 6: Meander for the $\epsilon = 0.75$ case. Note the spiral has completely unfolded, only two solutions are possible.

