Nonlinear Stability, Thermoelastic Contact, and the Barber Condition

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The behavior of a one-dimensional thermoelastic rod is modeled and analyzed. The rod is held fixed and at constant temperature at one end, while at the other end it is free to separate from or make contact with a rigid wall. At this free end we impose a pressure and gap-dependent thermal boundary condition. This condition, known as the Barber condition, couples the thermal and elastic problems. Such systems have previously been shown to undergo a bifurcation from a unique linearly stable steady-state solution to multiple steady-state solutions with alternating stability. Here, the system is studied using the asymptotic matching techniques of boundary layer theory to derive short-time, long-time, and uniform expansions. In this manner, the analysis is extended into the nonlinear regime and dynamic information about the history dependence and temporal evolution of the solution is obtained. [DOI: 10.1115/1.1345699]

1 Introduction

The analysis of thermal contact problems has revealed a wealth of interesting phenomena. Beginning with J. R. Barber in 1978 ([1]), who pointed out that the solution of such problems poses certain difficulties, and continuing to this day, numerous researchers have turned their attention to these problems. Barber observed that the classical assumption of perfect insulation during a separated phase and perfect thermal contact during contact led to models with solutions which were unacceptable on physical grounds. Introducing a pressure and temperature-dependent boundary condition, which would subsequently become known as the Barber condition, he allowed for a smooth transition between the insulated and perfect thermal contact states. Studying a linearized version of a thermal contact problem which included the Barber condition, he showed that the paradoxes inherent in simpler models could be avoided and physically relevant solutions recovered.

In 1980, Barber, Dundurs, and Comninou [2] investigated a thermal contact problem using the Barber condition in a one-dimensional model of a thermoelastic rod. Imposing a temperature gradient across the rod, they demonstrated that the system underwent a bifurcation from a unique linearly stable steady-state solution to multiple solutions with alternating stability as the magnitude of the thermal gradient was varied.

Since that time, various authors have explored the Barber condition and its implications for thermal contact problems ([3,4]). While such analyses have been extended to multiple materials ([3,5,6]), various geometries ([7,8]), and to numerical simulations ([4]), most theoretical work to date has relied upon linear stability theory. In a recent article ([9]) we developed a nonlinear theory which described the history dependence and dynamics of solutions near the bifurcation point for a simplified model of a one-dimensional thermoelastic rod. Our model did not, however, include the Barber condition. Since the Barber condition is much more physically realistic than the boundary condition used in ([9]), it is desirable to have a nonlinear theory for a model which incorporates the Barber condition. We carry out such an analysis here.

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While the model studied here differs from the model studied in ([9]), only in the use of the Barber condition, the method of analysis differs significantly. In particular, here we use the asymptotic matching techniques of boundary layer theory to derive short-time, long-time, and uniform asymptotic expansions of the solution. In our prior analysis we used the method of multiple scales, or two-timing, to accomplish similar goals. The switch in techniques is not merely a matter of taste. Rather, any attempt to apply multiple scale techniques to the model considered herein will soon encounter algebraic difficulties. That is, such an attempt becomes analytically intractable. However, as is shown, boundary layer theory may be applied with little difficulty. This not only allows us to carry out the analysis for the one-dimensional rod model with the Barber condition, but gives us hope that similar techniques will yield a nonlinear stability theory for more complicated multidimensional problems.

We begin in Section 2 by formulating the governing equations for our model. We make the standard assumption that quasi-static uncoupled thermoplasticity is valid and use the Signorini contact condition to capture periods of separation and contact. We impose the Barber condition on the thermal part of the problem, leaving the contact resistance function unspecified. A solution is constructed for the elastic problem and the system of governing equations is reduced to a nonlocal and nonlinear heat conduction problem. In Section 3, we impose physically realistic constraints on the contact resistance and develop a linear theory. We review the analysis due to Barber [2], and show that the system studied undergoes a bifurcation from a single linearly stable steady-state solution to multiple steady-state solutions. Finally, in Section 4, we study the behavior of our system near the bifurcation point. That is, we inquire as to what happens when the system is started nearby the now linearly unstable steady-state solution. Using asymptotic matching techniques, we incorporate the effect of stabilizing nonlinear terms into our theory and obtain information about the dynamics and history dependence of the solution. We show that as conjectured, the solution does indeed approach one of the stable solutions obtained in the linear theory.

2 Formulation of the Model

We consider a one-dimensional thermoelastic rod of length L suspended between two rigid walls as pictured in Fig. 1. We assume that the rod possesses constant thermal and elastic material properties, is homogeneous and isotropic, and that uncoupled quasi-static thermoelastic theory is valid. With these assumptions

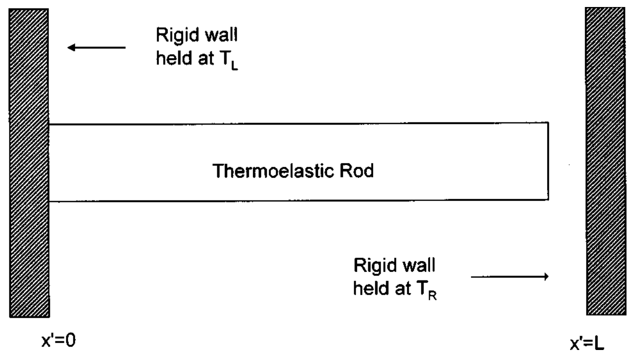

Fig. 1 Sketch of the model geometry

in mind, we formulate the equations governing the temperature distribution, T, elastic displacement, u', and stress, σ', within the rod. In the dimensionless variables

$$\theta=\frac{T-T_L}{T_R-T_L},\quad t=\frac{\kappa}{\rho c_p L^2}t',\quad x=\frac{x'}{L},\quad u=\frac{u'}{L},\quad \sigma=\frac{\sigma'}{E},\tag{2.1}$$

these equations take the form

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}\quad 0<x<1\tag{2.2}$$

$$\frac{\partial^2 u}{\partial x^2}=\mu\frac{\partial\theta}{\partial x}\quad 0<x<1\tag{2.3}$$

$$\sigma=\frac{\partial u}{\partial x}-\mu\theta\quad 0<x<1\tag{2.4}$$

$$\theta(0,t)=0\tag{2.5}$$

$$u(0,t)=0\tag{2.6}$$

$$\left\{\begin{array}{c} u\leqslant 0\\ \sigma\leqslant 0\\ u\sigma=0\end{array}\right\}\quad \text{at}\quad x=1\tag{2.7}$$

$$R(\eta)\frac{\partial\theta}{\partial x}(1,t)=1-\theta(1,t)\tag{2.8}$$

where here

$$\mu=\alpha(T_R-T_L),\quad R(\eta)=\frac{\kappa\hat{R}(\eta)}{L}\tag{2.9}$$

and

$$\eta=\sigma(1,t)-u(1,t).\tag{2.10}$$

Note that μ may be interpreted as a nondimensional coefficient of thermal expansion or as a dimensionless measure of a thermal gradient in the problem, while R is a dimensionless form of the contact resistance function. In fact, R may be thought of as a variable Biot number, measuring the relative strengths of heat conduction within the rod and ''convection'' through the rod's right end. The variable η is equal to the contact pressure during contact ($\eta<0$) and the gap size during periods of separation ($\eta>0$). For a full derivation of the model above, the reader is referred to ([9]). As mentioned in the Introduction, the model above differs from that in ([9]) in that the boundary condition at the right end of the rod in ([9]) is replaced here with the Barber condition, Eq. (2.8). The reader will also notice that in the above we have assumed a reference gap width of zero in Eq. (2.7).

Now, we note that the problems for u and σ are linear and may be solved exactly. That is, we may integrate Eq. (2.3) twice and use Eqs. (2.4), (2.6) and (2.7) to solve for u and σ. We find

$$u(x,t)=\mu\int_0^x\theta(\zeta,t)d\zeta-x\max\left\{\delta\int_0^1\theta(\zeta,t)d\zeta,0\right\}\tag{2.11}$$

and

$$\sigma(x,t)=-\max\left\{\mu\int_0^1\theta(\zeta,t)d\zeta,0\right\}.\tag{2.12}$$

Using these solutions, we may evaluate η, i.e.,

$$\eta=\sigma(1,t)-u(1,t)=-\mu\int_0^1\theta(\zeta,t)d\zeta.\tag{2.13}$$

Hence, we have reduced the problem to one for θ only. We are left with

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}\quad 0<x<1\tag{2.14}$$

$$\theta(0,t)=0\tag{2.15}$$

$$R(\eta)\frac{\partial\theta}{\partial x}(1,t)=1-\theta(1,t)\tag{2.16}$$

$$\eta=-\mu\int_0^1\theta(\zeta,t)d\zeta.\tag{2.17}$$

3 Linear Theory

In order to proceed with the analysis, we must further characterize the contact resistance function, $R(\eta)$. We recall from the definition of η, Eq. (2.10), that $\eta>0$ corresponds to separation from the wall, and that in this case η measures the size of this gap. Physically, we expect the contact resistance to increase monotonically with gap size. On the other hand, $\eta<0$ corresponds to contact with the wall, and in this case η measures the contact pressure. Here, we expect contact resistance to decrease monotonically with increasing pressure. Further, contact resistance must be a positive quantity and on physical grounds we are led to expect that $R(\eta)$ appears as pictured in Fig. 2.

With these assumptions about R in mind, we may investigate steady-state solutions of the system (2.14), (2.15), and (2.16). We begin by setting the time derivative to zero in Eq. (2.14), integrating the resulting ode and using the boundary conditions, Eqs. (2.15) and (2.16), to determine that steady solutions must have the form

$$\theta^*(x)=ax\tag{3.1}$$

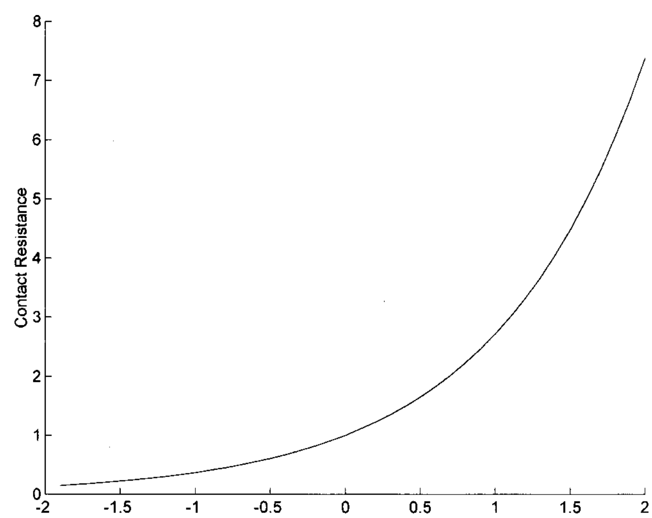

Fig. 2 A typical contact resistance function

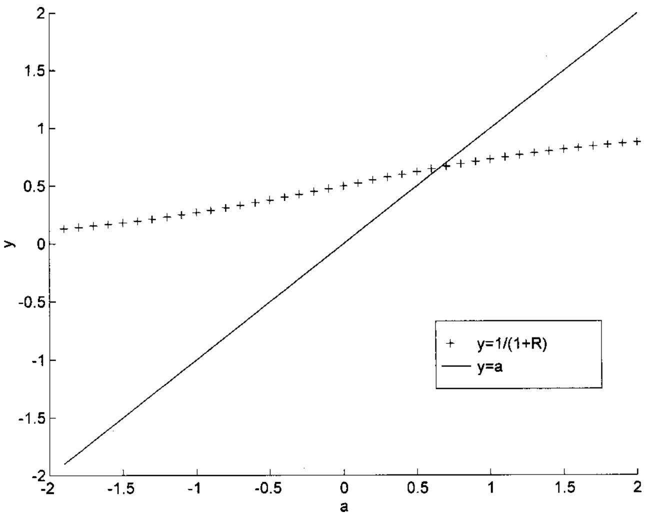

Fig. 3 Geometric solution of the steady-state problem

where a satisfies

$$a=\frac{1}{1+R(-\mu a/2)}. \tag{3.2}$$

Our observations about the nature of R allow us to plot the left and right sides of Eq. (3.2) on the same plot as functions of a. This is done in Fig. 3. With physically realistic assumptions on R, it is clear that we will always have at least one point of intersection, and hence at least one steady solution. We also note that depending upon the exact nature of R, we may have more than one steady-state solution. To clarify this situation further, we need more detailed information about the contact resistance. For simplicity, we specify the value of R at a convenient point. In particular, we shall assume that $R(-\mu/4)=1$. This implies that $a=1/2$ is a solution of Eq. (3.2) and hence $\theta^*(x)=x/2$ is a steady-state solution of the system (2.14), (2.15), and (2.16). Next, we define

$$F(\eta)=\frac{1}{1+R(\eta)} \tag{3.3}$$

and note that $F(-\mu/4)=1/2$. In order to have a bifurcation of the type investigated by Barber [2], it is easy to see that we must have that $\eta=-\mu/4$ be an inflection point for F. That is, we assume $F''(-\mu/4)=0$ and $F'''(-\mu/4)>0$. This implies that we may expand F in a Taylor series about $\eta=-\mu/4$ as follows:

$$F(\eta)=\frac{1}{2}+(\eta+\mu/4)F'(-\mu/4)+\frac{(\eta+\mu/4)^3}{6}F'''(-\mu/4)+\cdots. \tag{3.4}$$

Throughout the remainder of this paper we shall localize the analysis about the steady-state solution $\theta^*(x)=x/2$. That is, in addition to assuming that $\eta=-\mu/4$ is an inflection point for F, we assume that nearby this inflection point F is well approximated by the first three nonzero terms in the Taylor series (3.4).

Next, we investigate the linear stability of the steady solution $\theta^*(x)=x/2$. Accordingly we seek a solution to (2.14), (2.15), and (2.16) in the form

$$\theta(x,t)=\frac{x}{2}+\phi(x)e^{-\lambda^2 t} \tag{3.5}$$

where $|\phi(x)|\ll 1$. Inserting this ansatz into our governing equations, expanding the nonlinear terms in Taylor series, and omitting quadratic and higher order terms in ϕ, we obtain the eigenvalue problem

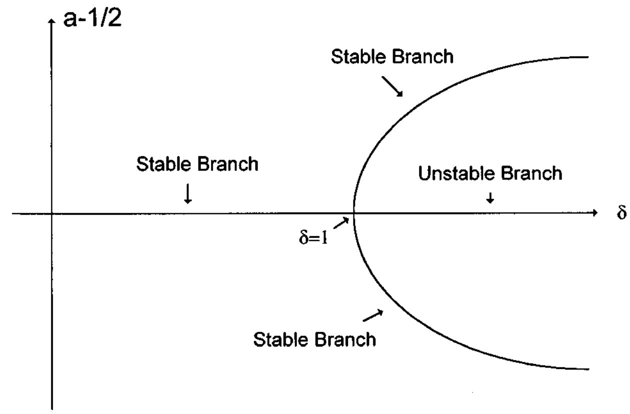

Fig. 4 Bifurcation diagram showing the constant in the steady solution as a function of the bifurcation parameter, δ

$$\frac{d^2\phi}{dx^2}+\lambda^2\phi=0 \tag{3.6}$$

$$\phi(0)=0 \tag{3.7}$$

$$\phi'(1)+\phi(1)=4\delta\int_0^1\phi(\zeta)d\zeta. \tag{3.8}$$

Here $\delta=\mu R'(-\mu/4)/8$. This linear eigenvalue problem has a solution ϕ when λ satisfies

$$\lambda^2\cos(\lambda)+4\delta(\cos(\lambda)-1)+\lambda\sin(\lambda)=0. \tag{3.9}$$

The solutions of this equation in conjunction with Eq. (3.5) determine the stability of the perturbation ϕ; if $\mathrm{Re}(\lambda^2)>0$ (<0), then the steady state is linearly stable (unstable). The solutions of Eq. (3.9) were studied by Barber [2]; we do not repeat his analysis here. Rather, we simply note that in our notation, Barber's result is that $\delta<1$ corresponds to linear stability, $\delta>1$ corresponds to linear instability, while $\delta=1$ is the marginally stable case.

With the assumptions mentioned above concerning F, another alternative characterization of the bifurcation as δ passes through one is possible. Retaining only up to cubic terms, and using (3.4) in the equation defining the steady states, (3.2), we obtain a cubic polynomial for a. By construction, one solution is of course, $a=1/2$. The other two solutions are given by

$$a=\frac{1}{2}\pm\left(\frac{2}{\mu}\right)^{3/2}\sqrt{\frac{6(\delta-1)}{F'''(-\mu/4)}}. \tag{3.10}$$

We see that these solutions are unphysical (imaginary) for $\delta<1$, and that we pick up two new physical solutions as δ passes through one. At least locally, the bifurcation is of the standard pitchfork type. The linear theory is summarized in Fig. 4.

4 Nonlinear Theory

In the previous section we found and investigated the linear stability of steady-state solutions to our model, i.e., Eqs. (2.14)–(2.16). We made physically realistic assumptions about the contact resistance function R, and determined that $\theta^*(x)=x/2$ was a solution for all positive values of the parameter δ. We showed that the linear stability of this solution changed as δ passed through one. In particular, for $\delta<1$, this solution was found to be linearly stable, while for $\delta>1$, linear theory predicts that any infinitesimal perturbation will grow exponentially. Clearly, in this parameter range, the linear theory is only valid for a limited time. In this section, we use the asymptotic matching techniques of boundary layer theory to extend our analysis into the nonlinear regime. That is, we investigate the nature of the solution to our governing equations when δ is nearby, but greater than one, and the initial conditions are such that the system starts near the now unstable solution, θ^*. Our goal is to develop an approximate solution which is valid for all time, thereby allowing us to understand the dynamics and history dependence of solutions near this bifurcation point.

We begin, by imposing the initial condition

$$\theta(x,0)=\frac{x}{2}+\epsilon h(x). \tag{4.1}$$

Here, $\epsilon \ll 1$ and $h(x)$ is an arbitrary $O(1)$ function. Note that this defines ϵ and starts our system near $\theta^*(x)$. It is now convenient to rescale by setting $\theta=\epsilon v+\theta^*(x)$. Introducing this rescaling into Eqs. (2.14), (2.15), (2.16), and (4.1), expanding the nonlinear terms in a Taylor series and retaining terms up to $O(\epsilon^2)$ we obtain

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2} \tag{4.2}$$

$$v(0,t)=0 \tag{4.3}$$

$$\frac{\partial v}{\partial x}(1,t)+v(1,t)=\left(4\delta\int_0^1 v(\zeta,t)d\zeta\right)\left[1+\epsilon\frac{\partial v}{\partial x}(1,t)-\epsilon v(1,t)\right]$$
$$-\epsilon^2 c_0^2\left(\int_0^1 v(\zeta,t)d\zeta\right)^3 \tag{4.4}$$

$$v(x,0)=h(x) \tag{4.5}$$

where here $c_0^2=\mu F'''/3$ and is, by assumption, a positive number. We assume $c_0^2=O(1)$.

Next, we let $\delta=1+\gamma\epsilon^2$ where $\gamma=O(1)$ and we seek a solution in the form

$$v(x,t)\sim v_0(x,t)+\epsilon v_1(x,t)+\epsilon^2 v_2(x,t)+\cdots. \tag{4.6}$$

Inserting this expansion into Eqs. (4.2)–(4.5), and equating to zero coefficients of powers of ϵ, we find that $v_0(x,t)$ satisfies

$$\frac{\partial v_0}{\partial t}=\frac{\partial^2 v_0}{\partial x^2} \tag{4.7}$$

$$v_0(0,t)=0 \tag{4.8}$$

$$\frac{\partial v_0}{\partial x}(1,t)+v_0(1,t)=4\int_0^1 v_0(\zeta,t)d\zeta \tag{4.9}$$

$$v_0(x,0)=h(x). \tag{4.10}$$

We construct a solution using eigenfunction expansion. Accordingly, we seek solutions in the form $A(t)\phi(x)$, separate variables and obtain the eigenvalue problem, Eqs. (3.6)–(3.8), with $\delta=1$ for the spatial eigenfunctions, $\phi(x)$. Hence the eigenvalues are given by Eq. (3.9) with $\delta=1$. Further, from the linear theory in the previous section and from Barber's analysis, we note that zero is an eigenvalue and that all other eigenvalues are purely real. Next, we must take our analysis one step further and explicitly construct the eigenfunctions and derive an expansion theorem. Towards this end it is useful to remove the integral from the boundary condition, Eq. (3.8). We integrate Eq. (3.6) from zero to one, solve for the integral, and use this result to eliminate the integral in Eq. (3.6). This yields the equivalent system

$$\frac{d^2\phi}{dx^2}+\lambda^2\phi=0 \tag{4.11}$$

$$\phi(0)=0 \tag{4.12}$$

$$4\left(\frac{d\phi}{dx}(0)-\frac{d\phi}{dx}(1)\right)=\lambda^2\left(\frac{d\phi}{dx}(1)+\phi(1)\right). \tag{4.13}$$

We note that $\lambda=0$ remains an eigenvalue of this system. However, the new formulation, (4.11)–(4.13), makes clear the fact that we are faced with a nonstandard eigenvalue problem. That is, the eigenvalue parameter, λ, appears in the boundary conditions. Consequently, we cannot simply consider the operator $L=-d^2/dx^2$ and rely upon the theory of eigenfunction expansion for $L\phi=-\lambda^2\phi$. Rather, we must exercise care in defining an operator, constructing an adjoint, and in deriving an expansion theorem. We follow a typical approach as outlined, for example, in Friedman [10].

We begin by considering the space of two component vectors U, whose first component is a real-valued C^2 function, $u(x)$, and whose second component is a real number, u_1. We define the inner product of two vectors in this space by

$$\langle U,V\rangle=\int_0^1 u(x)v(x)dx+u_1v_1. \tag{4.14}$$

Next, we restrict our attention to the subspace, D, of vectors U such that $u(0)=0$ and $u_1=u(1)+u'(1)$. Then, we define an operator L acting on elements U of D by

$$LU=\begin{pmatrix} -\dfrac{d^2u}{dx^2} \\ 4\left(\dfrac{du}{dx}(0)-\dfrac{du}{dx}(1)\right)\end{pmatrix}. \tag{4.15}$$

Note that our eigenvalue problem, Eqs. (4.11)–(4.13), is now simply stated as find a vector U in D such that $LU=\lambda^2U$. Further, we may define an adjoint operator, L^*, where

$$L^*V=\begin{pmatrix} -\dfrac{d^2u}{dx^2} \\ \dfrac{dv}{dx}(1)\end{pmatrix} \tag{4.16}$$

and acts on elements, V, of the subspace D^* defined as two component vectors satisfying $v'(1)=v(0)-v(1)$ and $v_1=-v(0)/4$. The reader may easily verify that with the inner product, (4.14), we have $\langle LU,V\rangle=\langle U,L^*V\rangle$.

Next, we may attempt to derive an expansion theorem and solve our leading order problem. First, we note, of course, that the discrete spectrum of L is given by Eq. (3.9). The fact that our operator is not self-adjoint raises the possibility that L also possesses a continuous spectrum which would effect the nature of an eigenfunction expansion. By using a Green's function approach, we may rule out this possibility. The details of obtaining this null result are lengthy, the interested reader is referred to Appendix B of ([9]) for an example of this calculation. This having been said, we now construct eigenvectors. We find

$$U_n=\begin{pmatrix} \dfrac{a_n\sin(\lambda_n x)}{\lambda_n} \\ \dfrac{a_n\lambda_n\cos(\lambda_n)+a_n\sin(\lambda_n)}{\lambda_n}\end{pmatrix} \tag{4.17}$$

where $\lambda_0=0$ and the remaining λ_n's are the real nonzero solutions of Eq. (3.9) for $\delta=1$. Similarly, we can construct the following adjoint eigenfunctions from our adjoint eigenvalue problem:

$$V_n=\begin{pmatrix} b_n\left(\cos(\lambda_n x)+\left(\dfrac{4\sin(\lambda_n)-\lambda_n}{4\cos(\lambda_n)}\right)\sin(\lambda_n x)\right) \\ -\dfrac{b_n}{4}\end{pmatrix} \tag{4.18}$$

where, of course, the λ_n's are the same as above. We note that $\langle U_n,V_m\rangle=0$ for all $n\neq m$ and that $\langle U_n,V_n\rangle\neq 0$ for all $n\neq 0$ while $\langle U_0,V_0\rangle=0$. Further, we choose the a_n's, for $n\neq 0$ so that $\langle U_n,V_n\rangle=1$ and we choose a_0 so that $\langle U_0,U_0\rangle=1$.

Now, we can construct a solution to our leading order problem by using the eigenvectors and the adjoint eigenvectors just defined. We seek a solution lying in the subspace D in the following form:

$$\begin{pmatrix} v_0(x,t) \\ \frac{\partial v_0}{\partial x}(1,t)+v_0(1,t) \end{pmatrix} = \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 t} U_n . \tag{4.19}$$

The governing equation and the boundary conditions are of course satisfied, while the A_n's are still unknown. They will be determined by our initial conditions. We require that

$$\begin{pmatrix} v_0(x,0) \\ \frac{\partial v_0}{\partial x}(1,0)+v_0(1,0) \end{pmatrix} = \begin{pmatrix} h(x) \\ h_1 \end{pmatrix} = \sum_{n=0}^{\infty} A_n U_n \tag{4.20}$$

where h_1 and the A_n are yet to be determined. If we take an inner product with V_m, where $m \neq 0$, we find $A_m = \langle H, V_m \rangle$ where

$$H = \begin{pmatrix} h(x) \\ h_1 \end{pmatrix}. \tag{4.21}$$

Next, we take an inner product with V_0 and find $\langle H, V_0 \rangle = 0$ which implies

$$4\int_0^1 h(x)dx = h_1 \tag{4.22}$$

and hence uniquely determines h_1. Now, A_0 is still undetermined. To remedy this situation, we take an inner product with U_0 across Eq. (4.20) and solve for A_0 to find

$$A_0 = \langle H, U_0 \rangle - \sum_{n=1}^{\infty} A_n \langle U_n, U_0 \rangle. \tag{4.23}$$

We now have a complete solution for $v_0(x,t)$.

Noting that $\lambda_0 = 0$ and that all other λ_n's are real, we see from Eq. (4.19) that all modes except for the λ_0 mode decay in the large time limit. This implies that as $t \to \infty$ we have $v_0(x,t) \to A_0\sqrt{3/13}x$. If we now attempted to compute a solution for $v_1(x,t)$, which is forced by the v_0 solution, we would find that $v_1 \to \infty$ as $t \to \infty$! This implies that our expansion is nonuniform in time, and hence only serves as a *short-time* solution. To obtain the long-time behavior of our system, we turn to boundary layer theory.

We begin by changing to the slow or long time scale $\tau = \epsilon^2 t$. Our problem for v, Eqs. (4.2)–(4.5), becomes

$$\epsilon^2 \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} \tag{4.24}$$

$$v(0,\tau) = 0 \tag{4.25}$$

$$\frac{\partial v}{\partial x}(1,\tau) + v(1,\tau) = \left(4\delta \int_0^1 v(\zeta,t)d\zeta\right)\left[1+\epsilon\frac{\partial v}{\partial x}(1,\tau) - \epsilon v(1,\tau)\right] - \epsilon^2 c_0^2 \left(\int_0^1 v(\zeta,\tau)d\zeta\right)^3. \tag{4.26}$$

Here, we seek a solution in the form

$$v(x,\tau) \sim v_0(x,\tau) + \epsilon v_1(x,\tau) + \epsilon^2 v_2(x,\tau) + \cdots. \tag{4.27}$$

Introducing this expansion into our long-time Eqs. (4.24)–(4.26), and equating to zero coefficients of powers of ϵ we again obtain an infinite set of equations which sequentially determine the v_n. In order to determine the leading order solution, we shall need the equations up to order ϵ^2. Our order one equations are

$$\frac{\partial^2 v_0}{\partial x^2} = 0 \tag{4.28}$$

$$v_0(0,\tau) = 0 \tag{4.29}$$

$$\frac{\partial v_0}{\partial x}(1,\tau) + v_0(1,\tau) = 4\int_0^1 v_0(\zeta,\tau)d\zeta. \tag{4.30}$$

This system may be solved and we find $v_0(x,\tau) = A(\tau)x$ where $A(\tau)$ is an undetermined function of the slow time variable, τ. Using this we may simplify the $O(\epsilon)$ system and we find

$$\frac{\partial^2 v_1}{\partial x^2} = 0 \tag{4.31}$$

$$v_1(0,\tau) = 0 \tag{4.32}$$

$$\frac{\partial v_0}{\partial x}(1,\tau) + v_0(1,\tau) = 4\int_0^1 v_1(\zeta,\tau)d\zeta. \tag{4.33}$$

This system may also be solved and we find $v_1(x,\tau) = B(\tau)x$ where B is an unknown function. Using our $O(1)$ and $O(\epsilon)$ solutions we can simplify the $O(\epsilon^2)$ problem. We find that $v_2(x,\tau)$ satisfies

$$\frac{\partial v_0}{\partial \tau} = \frac{\partial^2 v_2}{\partial x^2} \tag{4.34}$$

$$v_2(0,\tau) = 0 \tag{4.35}$$

$$\frac{\partial v_2}{\partial x}(1,\tau) + v_2(1,\tau) = 4\int_0^1 v_2(\zeta,\tau)d\zeta + 4\gamma\int_0^1 v_0(\zeta,\tau)d\zeta - c_0^2\left(\int_0^1 v_0(\zeta,\tau)d\zeta\right)^3. \tag{4.36}$$

Using our solution for v_0, integrating Eq. (4.34) with respect to x and applying the boundary conditions, we find that this system only possesses a solution if $A(\tau)$ satisfies

$$\frac{dA}{d\tau} = 4\gamma A - \frac{c_0^2}{4}A^3. \tag{4.37}$$

This ordinary differential equation determines $A(\tau)$ up to an arbitrary constant which is obtained by matching back to the short-time solution. That is the initial condition for $A(\tau)$ is given by

$$A(0) = \lim_{t\to\infty} \frac{v_0(x,t)}{x} = A_0\sqrt{\frac{3}{13}}. \tag{4.38}$$

Our short-time solution, Eq. (4.19), and our long-time solution, $A(\tau)x$, may be assembled into a uniformly valid solution. That is

$$v(x,t) \sim A(\epsilon^2 t)x - A_0\sqrt{\frac{3}{13}}x + \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 t}\frac{a_n \sin(\lambda_n x)}{\lambda_n} + O(\epsilon) \tag{4.39}$$

gives the leading order behavior of solutions for all time.

5 Discussion

We began by formulating a model of a one-dimensional thermoelastic rod subjected to conditions which allowed for thermoelastic contact and the possibility of a thermoelastic instability. In contrast to our earlier nonlinear stability theory, ([9]), in this model we included a general form of the Barber condition. Physically based assumptions about the nature of the contact resistance function, $R(\eta)$, were made. With these assumptions we set out to verify Barber's linear theory ([2]), and to extend his analysis into the nonlinear regime.

The linear theory showed that for a certain class of contact resistance functions, or more precisely for suitable assumptions on the reciprocal contact resistance function, F, the system underwent a bifurcation as δ passed through one. That is, just as Barber discovered, there is a transition from one stable stationary solution

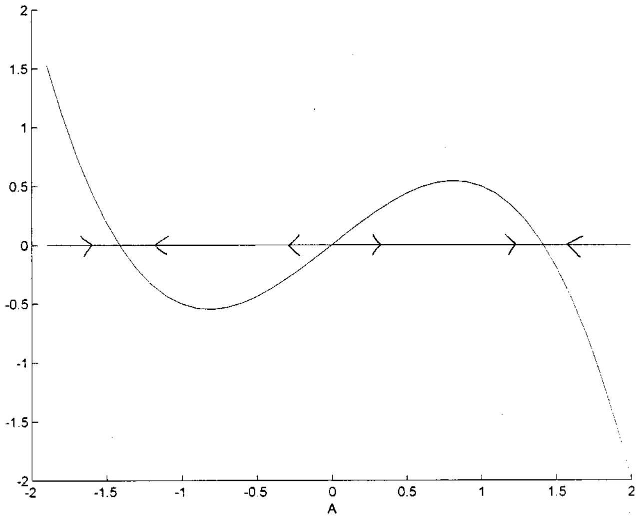

Fig. 5 Behavior of solutions to the amplitude equation, which governs $A(\tau)$

to three solutions with alternating stability. The solution that was stable undergoes an exchange of stabilities and becomes unstable for $\delta>1$.

Next, we attempted to extend the linear analysis into the nonlinear regime in the neighborhood of $\delta=1$. In doing so, we hoped to verify the conjecture of Barber that solutions which start near the now unstable steady-state approach one of the two linearly stable solutions uncovered in the linear analysis. Further, we would like the nonlinear analysis to clarify what initial conditions go to which solution and how they get there. That is, we want to understand history dependence and dynamics in the neighborhood of the bifurcation. To accomplish this goal, we developed a uniformly valid asymptotic approximation to the solution using the asymptotic matching techniques of boundary layer theory. These techniques yielded the following asymptotic approximation to the solution:

$$\theta(x,t)\sim\frac{x}{2}+\epsilon A(\epsilon^2 t)x-\epsilon A_0\sqrt{\frac{3}{13}}x+\epsilon\sum_{n=0}^{\infty}A_n e^{-\lambda_n^2 t}\frac{a_n\sin(\lambda_n x)}{\lambda_n}+O(\epsilon^2) \tag{5.1}$$

where $A(\tau)$ satisfied the amplitude equation

$$\frac{dA}{d\tau}=4\gamma A-\frac{c_0^2}{4}A^3 \tag{5.2}$$

with initial condition

$$A(0)=A_0\sqrt{\frac{3}{13}}. \tag{5.3}$$

Now, as time tends to infinity all contributions from the sum in Eq. (5.1) decay to zero or cancel with other terms. This means that the limiting behavior is given by

$$\lim_{t\to\infty}\theta(x,t)\sim\frac{x}{2}+\epsilon x\lim_{t\to\infty}A(\epsilon^2 t)+O(\epsilon^2) \tag{5.4}$$

and our questions concerning nonlinear stability may be answered by examining the governing equation for A, i.e., Eq. (5.2). We note that Eq. (5.2) includes a cubic nonlinearity. Recall that this term arose due to the nonlinear nature of the contact resistance function, $R(\eta)$. Further note that this nonlinearity exerts a stabilizing influence on the solution. In Fig. 5, we sketch the phase plane for this amplitude equation. We see that A approaches $\pm 4\sqrt{\gamma}/c_0$ according as the initial condition is positive or negative. This implies that the solution tends to ax where

$$a=\frac{1}{2}\pm\left(\frac{2}{\mu}\right)^{3/2}\sqrt{\frac{6(\delta-1)}{F'''(-\mu/4)}}. \tag{5.5}$$

These are precisely the solutions uncovered by the linear theory and hence Barber's conjecture is verified. Further, questions of history dependence may now be answered by simply examining the initial condition on A. The sign of this condition dictates whether we tend to the positive or negative solution. This sign in turn simply depends on the direction of the perturbation to the unstable steady solution. Similarly, questions concerning dynamics of solutions are answered by the time behavior of Eqs. (5.1) and (5.2).

Finally, a comment about the method of analysis is in order. As stated in the Introduction, the switch from the method of multiple scales to boundary layer theory was necessary in order to be able to carry out the analysis. As can be seen from the section on the nonlinear theory, this technique allows one to explicitly solve the reduced equations at each order. Such solutions are algebraically intractable with the multiscale approach. This phenomena has been observed in nonlinear stability theory for other types of problems, examples may be found in ([11]) or ([12]). This simplicity does, however, come at a price. In particular, we only discover the slow time behavior of the *dominant* mode. As all other modes decay, this price is not too steep, but yet it should be acknowledged. The gift of simplicity, however, gives one hope that multidimensional nonlinear stability theories are within reach. As a final note of inspiration to the reader, we observe that Barber's linear theory has now been extended to a nonlinear theory in a neighborhood of the bifurcation. Such a nonlinear theory is often referred to as a weakly nonlinear stability theory. A global nonlinear stability theory would be of interest and remains a challenge for the curious researcher.

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