

# Lenz's Law and Dimensional Analysis

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We show that the flight time of a magnet through a non-magnetic metallic tube may be analyzed via dimensional analysis. This simple result makes this classic, dramatic demonstration of Lenz's law accessible qualitatively and quantitatively to students with little knowledge of electromagnetism and only elementary knowledge of calculus. In addition, this exercise provides a new example of the power and limitations of dimensional analysis.

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## I. INTRODUCTION

The availability of small, strong, inexpensive neodymium-iron-boron magnets led to the development of the elegant “slowly falling magnet” classroom demonstration of Lenz's Law. In this experiment a powerful magnet is dropped down a nonmagnetic conducting tube. Eddy currents induced in the pipe by the falling magnet set up a field opposing the motion of the magnet. The resulting drag rapidly brings the magnet to terminal velocity, and the fall through the pipe takes up to ten times longer than for an otherwise identical nonmagnetic object dropped through the same tube. This demonstration, the related demonstration of magnetic damping of a pendulum, and the application of this phenomena to magnetic braking has led to a flurry of fascinating articles on eddy current damping.<sup>1-5</sup>

To date, analysis of this experiment has either involved fitting parameters in a force balance model to data<sup>1,3</sup> or performing a first principles study via Maxwell's equations.<sup>2,4,5</sup> In the first case, the analysis remains somewhat unconvincing to the inquisitive student who asks what happens when different size magnets or different size pipes are used. The model fails to be suitably predictive. In the second case, while magnetic, geometric, and material effects are accounted for, the analysis is complicated and requires the student to understand Maxwell's equations. In this paper, we show that a middle road is possible. In particular, we combine a force balance model with the powerful tool of *dimensional analysis* in order to obtain a model that is both simple and predictive. In addition to providing an interesting new application of dimensional analysis, this paper makes the quantitative study of this elegant demonstration accessible to a wider audience. The problem can even be posed as an engineering design problem for students with no more than a working knowledge of elementary calculus. The question of how flight time of the magnet through the pipe varies with pipe radius, pipe conductivity, and magnet strength can be addressed in a quantitative way.

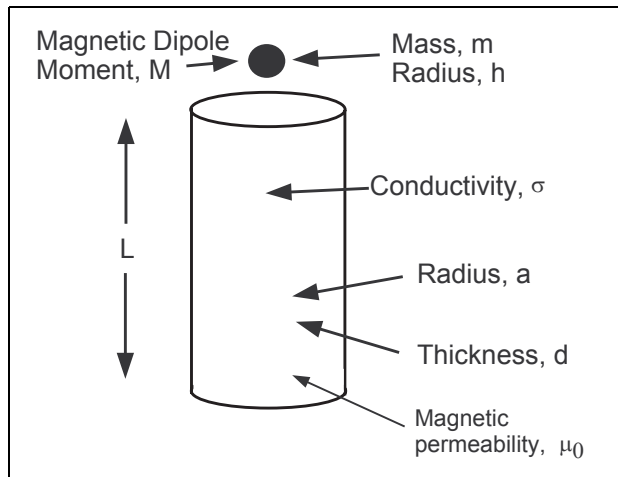


FIG. 1: The classic Lenz's law demonstration.

## II. ANALYZING THE DEMONSTRATION

Since both magnets and metal pipes are inexpensive one is likely to have various size pipes, pipes of various materials, and an assortment of magnets lying around. Going beyond simply performing the falling magnet demonstration and encouraging students to “play” with these materials rapidly leads to the observation that the time of flight of the magnet depends on pipe radius, pipe material, and magnet strength. If the students are familiar with Maxwell's equations this provides a wonderful opportunity to discuss the analysis in the classic paper of Saslow.<sup>2</sup> Students at a more elementary level (or from a different background) can be introduced to and encouraged to attempt a dimensional analysis in order to predict the dependence of time of flight on pipe radius and other parameters.<sup>7</sup>

A first attempt at a dimensional analysis might proceed by drawing a sketch similar to Figure 1, identifying important parameters, and then assuming a dependence of the time of flight,  $t'_f$ , in the form

$$t'_f = F(L, a, h, d, m, g, M, \mu_0, \sigma). \quad (1)$$

Such an analysis fails miserably! The system is sim-

ply underdetermined and no reasonable inferences about functional dependence may be made. The reader is encouraged to try this for himself. Instead what is called for here, is the combination of elementary physics and dimensional arguments. Applying Newton's second law to the falling magnet of Figure 1 leads to

$$m\ddot{x} = -mg - k\dot{x} \quad (2)$$

$$x(0) = L, \quad \dot{x}(0) = 0. \quad (3)$$

Here, dots indicate differentiation with respect to time,  $t'$ ,  $g$  is the gravitational acceleration, and  $k$  is an unknown constant measuring the strength of magnetic damping. The position of the magnet is measured upwards from the bottom of the pipe. While the first two terms in equation (2), i.e., the inertial term and the gravitational acceleration, will be convincing to the student, the third term requires some argument. Why is a linear drag term appropriate? At this point the instructor may discuss Faraday's law, indicating that the induced electromotive force is proportional to the time rate of change of the magnetic field. In this case, the time rate of change of the magnetic field is essentially the same as the velocity of the magnet, hence drag should be proportional to velocity.

It is instructive to recast equations (2)-(3) in dimensionless form. Introducing the rescaling

$$u = \frac{x}{L}, \quad t = \frac{mg}{kL}t' \quad (4)$$

yields

$$\frac{m^2g}{k^2L} \frac{d^2u}{dt'^2} = -1 - \frac{du}{dt'} \quad (5)$$

$$u(0) = 1, \quad \frac{du}{dt'}(0) = 0. \quad (6)$$

The student may wonder why the "standard" inertial time scale was not chosen. This provides the instructor with the opportunity to explain that the choice of time scale depends on the parameter range of interest. From observation of the experiment we see that inertial forces act over a very short time while magnetic damping acts over most of the lifetime of the experiment. Since our aim is to deduce the time of flight, we focus on the damping effect by choosing a damping time scale. This observation further implies that the dimensionless parameter

$$\epsilon = \frac{m^2g}{k^2L} \quad (7)$$

is in fact a *small* parameter. That is,  $\epsilon \ll 1$ . To simplify the analysis further we set  $\epsilon$  to zero and solve for  $u(t)$  to find

$$u(t) = 1 - t. \quad (8)$$

We note that the initial condition  $u'(0) = 0$  is not satisfied, we are in fact ignoring the small time interval over which the magnet velocity goes from zero to terminal.<sup>8</sup> From equation (8) we see that the dimensionless time of

flight,  $t_f$ , equals one, and the dimensional flight time is given by

$$t'_f = \frac{kL}{mg}. \quad (9)$$

While this is physically reasonable and may be used to fit data as in Priest and Wade<sup>1</sup>, this result gives no insight into the flight time's dependence on pipe radius, etc. This result only gives the dependence on gravity, mass, and pipe length. Effects due to magnet strength, pipe radius, etc., are only captured by the unknown damping constant  $k$ . Here, we may fruitfully employ dimensional analysis. Instead of trying to uncover the dependence of time of flight on various parameters, we may attempt to uncover the functional dependence of the unknown damping constant,  $k$ , on key parameters. We assume a dependence

$$k = f(\mu_0, M, \sigma, a). \quad (10)$$

Note that we have greatly reduced the number of parameters over equation (2). It is clear the damping constant should not depend on pipe length,  $L$ , magnet mass,  $m$ , or gravitational acceleration,  $g$ . Hence these need not appear on the right hand side of equation (10). We do however, still have three possible choices of length scales to include,  $a$ ,  $d$ , and  $h$ . We choose the pipe radius,  $a$ , as our length scale in equation (10). We ignore the pipe thickness,  $d$ , and the magnet radius,  $h$ . One may argue that the pipe thickness effect is not distinct from the conductivity,  $\sigma$ , and that the magnet radius effect is not distinct from the magnetic permeability,  $\mu_0$ . Now, we form a dimensionless combination of  $k$  and our parameters and set it equal to a dimensionless function of the parameters

$$\frac{ka^4}{\mu_0^2\sigma M} = \phi(\mu_0, M, \sigma, a). \quad (11)$$

Making the standard hypothesis that the argument of  $\phi$  is a product of the powers of the parameters, and examining units, we quickly uncover the fact that  $\phi$  must be a dimensionless constant. Hence

$$k = c \frac{\mu_0^2\sigma M}{a^4} \quad (12)$$

and the time of flight is given by

$$t'_f = c \frac{\mu_0^2\sigma ML}{mga^4}. \quad (13)$$

We note that this agrees with the analysis of Saslow,<sup>2</sup> modulo the unknown dimensionless constant  $c$ . Of course, now, a *single* experiment fixes the value of  $c$ , and equation (13) becomes predictive. Moreover, it is now easy to do comparative quantitative experiments with only a ruler and a stopwatch as measurement tools. For example, if we use one magnet and two copper pipes

Pipe 1	Pipe 2
1.40	0.46
1.34	0.45
1.36	0.45
1.34	0.27
1.42	0.31
1.30	0.43
1.34	0.43
1.40	0.28
1.33	0.39
1.36	0.28

TABLE I: A sample set of flight times measured in seconds. Pipe 1 was diameter 2.3cm. Pipe 2 was diameter 1.43cm.

which differ only in radius, we may write the ratio of flight times as

$$\frac{t'_{f1}}{t'_{f2}} = \frac{a_2^4}{a_1^4}. \quad (14)$$

Here subscripts denote pipe one and pipe two respectively. Note this ratio only depends on the radius of the pipes and hence this comparative experiment may be performed without knowledge of pipe conductivity, magnetic permeability, and the other parameters in the problem. Similarly, if two pipes with the same geometry but with different material properties are used, the ratio of flight times reduces to a ratio of conductivities.

### III. EXPERIMENT

To verify that the analysis introduced here could be easily incorporated into a classroom demonstration in a quantitative way, we designed and tested a modified version of the “slowly falling magnet” experiment. We limited our equipment to two copper pipes, a stopwatch, a spherical neodymium-iron-boron magnet, and a ruler. All supplies were readily available at local stores. The pipes were both 90cm long, the first had diameter  $a_1 = 1.43\text{cm}$ , while the second had diameter  $a_2 = 2.03\text{cm}$ . In each pipe, we drilled a 0.5cm hole at a distance 35.5cm from the top. The experimental procedure was to clamp

the pipe vertically to a stable surface such as a table or desk and to have one person drop the magnet into the top of the pipe while the second person started the stopwatch when the magnet passed the hole and stopped the stopwatch when the magnet exited the pipe. Measuring the time of flight in this manner, i.e., allowing the magnet to achieve its terminal velocity before starting the measurement, assured greater agreement with the small  $\epsilon$  approximation made earlier. A sample set of ten measurements of time of flight (in seconds) is shown in table I. The average flight time through pipe one was 1.359sec and the average flight time through pipe two was 0.375sec. According to equation 14, we should have  $t_1 \approx 4.061t_2$ . The measured average flight times puts  $t_1$  within 11% of its predicted value. This is remarkable agreement considering the hand-eye-stopwatch measuring error. A simple refinement of this experiment would be to drill the hole entirely through the pipe and use a photogate setup to measure flight times. Whichever approach one chooses, the experiment provides a convincing verification of the theory.

### IV. CLOSING COMMENTS

In a recent article, Price<sup>6</sup> expounds on the utility of dimensional analysis. In this article we have provided a new, novel example of the use of dimensional analysis, one that is easily coupled with experiment and makes a dramatic demonstration accessible to a wider audience. The analysis suggests numerous extensions and classroom projects. For example, students can quantitatively study weak vs. strong magnets, copper vs. aluminum pipes, or even spherical vs. disk shaped magnets.

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<sup>7</sup> For an introduction to dimensional analysis see Price<sup>6</sup>.

<sup>8</sup> In fact, there is a boundary layer in time in this problem. The inertial effect may be recovered by a boundary layer analysis. Alternately, the equation for  $u(t)$  may be solved exactly.