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To cite this article: R.P. Gilbert, Miao-Jung Ou & Yongzhi S. Xu (2001): The unknown object problem in a sea with sloping bottom, Applicable Analysis: An International Journal, 78:3-4, 497-511

To link to this article: http://dx.doi.org/10.1080/00036810108840947

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The Unknown Object Problem in a Sea with Sloping Bottom

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Communicated by C. Bu

(Received 25 November 2000)

This article considers the acoustic unknown object problem for a shallow ocean with a sloping seabed. The incident waves are sent from point sources along a straight line parallel to the sea surface, and the corresponding scattered fields are measured from a line above the unknown object. We prove a uniqueness theorem for the inverse problem, and describe a generalized dual space indicator method for numerical solution. Numerical results are given in Section 4.

Keywords: Inverse problems; Underwater acoustics; Undetermined object problem

AMS: 35J05; 86A35; 78A45

1 INTRODUCTION

Inverse, acoustic scattering problems in shallow ocean have been investigated by Gilbert, Xu and their coworkers in a series of papers. (See, e.g. [1,4,5,6–10,13,14].) In this paper we generalize these methods to oceans with sloping seabeds. To simplify the discussion we consider only two-dimensional oceans. For this case we are able to prove that

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the undetermined object problem has a unique solution. We denote the region of the ocean where we have a sloping seabed by the wedge
\[ R^2_\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq x_1 \tan(\theta_0), \quad 0 \leq x_1 \leq \infty \}, \]
or in polar coordinates as
\[ R^2_\Omega := \{(r, \theta) : 0 \leq r < \infty, \quad 0 \leq \theta \leq \theta_0 \}. \]
Here \( r = \sqrt{x_1^2 + x_2^2} \) and \( \theta = \tan^{-1}(x_2/x_1) \), and \( \theta_0 > 0 \) is the angle of the seabed; \( x_2 = 0 \) corresponds to the ocean surface. We designate the inclusion in the sea wedge by \( \Omega \), a bounded obstacle. The total time-harmonic acoustic field \( u \) generated by a point source at \( x^s = (x_1^s, x_2^s) \) satisfies the nonhomogeneous Helmholtz equation
\[
\Delta u(x) + k^2 u(x) = \delta(x - x^s), \quad \text{for } x = (x_1, x_2) \in R^2_\Omega \setminus \tilde{\Omega},
\]
\[ u = 0 \quad \text{at} \quad x_2 = 0, \]
\[ \frac{\partial u}{\partial n} = 0, \quad \text{at} \quad x_2 = x_1 \tan(\theta_0). \]
\[ u \] satisfies also the outgoing radiation condition
\[ \lim_{|x_1| \to \infty} \sqrt{r} \left( \frac{\partial u}{\partial r} - iku \right) = 0, \quad \text{for} \quad 0 < \theta < \theta_0, \]
where \( k > 0 \) is the wavenumber of the signal.

The unknown object \( \Omega \) may be in the middle of the sea wedge, or attached to the sea bottom (the sea mount problem). On the boundary of \( \Omega \), denoted by \( \partial \Omega \), \( u \) satisfies some unknown boundary condition such as a Dirichlet, Neumann or an impedance condition. We denote this condition by
\[ Bu = 0 \text{ on } \partial \Omega. \]

We consider the total wave \( u \) as the combination of an incident field \( u' \) plus a scattered field \( u'' \), \( u = u' + u'' \). As the incident field we take \( u' \) to be the Green's function for the sea wedge without any obstacle within.

From (1.1)–(1.4), we have
\[ \Delta u'(x) + k^2 u'(x) = 0, \quad \text{for } x \in \mathbb{R}^2 \backslash \tilde{\Omega} \] (1.6)

\[ u' = 0 \quad \text{at} \quad x_2 = 0, \] (1.7)

\[ \frac{\partial u'}{\partial n} = 0, \quad \text{at} \quad x_2 = x_1 \tan(\theta_0), \] (1.8)

\[ \lim_{|x_2| \to \infty} \sqrt{r} \left( \frac{\partial u'}{\partial r} - ik u' \right) = 0, \quad \text{for } 0 < \theta < \theta_0, \] (1.9)

On \( \partial \Omega \), \( u' \) satisfies some unknown boundary condition

\[ Bu' = -Bu' \quad \text{on} \quad \partial \Omega. \] (1.10)

The boundary conditions may therefore be any of the following:

Dirichlet boundary conditions:

\[ u' = -u' \quad \text{on} \quad \partial \Omega \] (1.11)

Neumann boundary conditions:

\[ \frac{\partial u'}{\partial n} = -\frac{\partial u'}{\partial n} \quad \text{on} \quad \partial \Omega \] (1.12)

Impedance boundary conditions:

\[ \frac{\partial u'}{\partial n} + \lambda u' = \frac{\partial u'}{\partial n} - \lambda u' \quad \text{on} \quad \partial \Omega \] (1.13)

where \( \text{Im} \lambda \geq 0 \). We assume that \( \partial \Omega \) is regular enough to admit a solution. The uniqueness of the direct scattering problem in the wedge can be proved in a way similar to that used in the proof for the homogeneous whole space case. From this it follows that for each given \( x' \), \( u'(x') = G(x, x') \) is known and \( u'(x) \) is determined uniquely for any given boundary condition \( B \).

Let

\[ \Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2_\theta \mid x_1 = x_1^0 \text{ constant}\}. \] (1.14)

\[ \Gamma_2 = \{(x_1, x_2) \in \mathbb{R}^2_\theta \mid x_2 = x_2^0 \text{ constant}\}. \] (1.15)
We assume that $\Gamma_2$ is 'above' the obstacle $\Omega$; i.e., $x_2^0 < \min\{x_2 \mid x \in \Omega\}$.
(Note that we denote the ocean surface by $x_2 = 0$.)

The inverse problem we consider is the following:

(I1): Given $u'(x, x')$ for $x$ and $x' \in \Gamma_1$, find the boundary of the unknown obstacle $\Omega$ without knowing which of the three boundary conditions $u'$ satisfies.

(I2): Given $u'(x, x')$ for $x$ and $x' \in \Gamma_2$, find the boundary of the unknown obstacle $\Omega$ without knowing which of the three boundary conditions $u'$ satisfies.

2 A UNIQUENESS THEOREM

In this section we prove an uniqueness theorem for the inverse problem described in section 1. Uniqueness results for obstacle inverse scattering problems in homogeneous space may be found in [11]. Similar results for parallel waveguide is given in [13].

Lemma 2.1 (Rellich's type Lemma) Let $u'$ be a solution of the Helmholtz equation in the exterior of $\Omega$ in $\mathbb{R}_2^n$ satisfying the boundary conditions (1.7), (1.8) and the outgoing radiation condition (1.9). If $u' = 0$ on $\Gamma_1$ or on $\Gamma_2$, then $u' = 0$ in $\mathbb{R}_2^n \setminus \Omega$.

Proof We first prove that if $u' = 0$ on $\Gamma_2$, then $u' = 0$ in $\mathbb{R}_2^n \setminus \Omega$. By the assumption, $u'$ satisfies the Helmholtz equation in the region between $x_2 = 0$ and $x_2 = x_2^0$:

$$D_2 := \{(x_1, x_2) \in \mathbb{R}_2^n \mid 0 < x_2 < x_1 \tan(\theta_0), \text{ if } 0 < x_1 < x_2^0 \cot(\theta_0);$$

$$0 < x_2 < x_2^0 \text{ if } x_1 > x_2^0 \cot(\theta_0)\},$$

and $u' = 0$ on $\{(x_1, x_2) \in \mathbb{R}_2^n \mid x_2 = x_2^0\}$ and when $x_2 = 0$; moreover, $\partial u'/\partial n = 0$ when $x_2 = x_1 \tan(\theta_0)$, and $u'$ satisfies the outgoing radiation condition as $r \to \infty$.

Introduce the auxiliary variables $\xi := x_1 - x_2 / \tan(\theta_0), \ \zeta := x_2 / \sin(\theta_0)$; then the inverse transformation is $x_1 = \xi - \zeta \cos(\theta_0), x_2 = \zeta \sin(\theta_0)$. The function $U'(\xi, \zeta) := u'(\xi - \zeta \cos(\theta_0), \zeta \sin(\theta_0))$ satisfies...
\[(1 + \cot^2(\theta_0)) U_{\xi \xi}^\prime + \csc^2(\theta_0) U_{\zeta \zeta}^\prime - \frac{2 \cos(\theta_0)}{\sin(\theta_0)} U_{\zeta}^\prime + k^2 U^\prime(x) = 0, \quad (2.16)\]

for \(0 < \xi\), and \(0 < \zeta < z_\delta / \sin(\theta_0)\).

\[
b / U^\prime = 0 \text{ at } \zeta = 0, \quad \text{and at } \zeta = z_\delta / \sin(\theta_0) , \quad (2.17)\]

\[
\frac{\partial U^\prime}{\partial (\xi - \xi)} = 0, \quad \text{at } \xi = 0, \quad (2.18)\]

\[
\lim_{\xi \to \infty} b / U^\prime = 0. \quad (2.19)\]

From (2.2) the function \(U^\prime\) has a Fourier expansion

\[
U^\prime(\xi, \zeta) = \sum_{n=1}^{\infty} u_n(\xi) \sin \left( \frac{n\pi \xi}{L} \right), \quad \text{where } L = z_\delta / \sin(\theta_0). \quad (2.20)\]

From (2.1), \(u_n(\xi)\) satisfies

\[
\sum_{n=1}^{\infty} \left\{ \left[ (1 + \cot^2(\theta_0)) u_n^\prime(\xi) + \left( \csc^2(\theta_0) \left( \frac{n\pi \xi}{L} \right)^2 + k^2 \right) u_n(\xi) \right] \sin \left( \frac{n\pi \xi}{L} \right) \right. \]

\[
- \frac{2 \cos(\theta_0)}{\sin(\theta_0)} \left( \frac{n\pi \xi}{L} \right) u_n^\prime(\xi) \cos \left( \frac{n\pi \xi}{L} \right) \right\} = 0, \quad \text{for } 0 < \xi \quad \text{and } 0 < \zeta < L. \quad (2.21)\]

It follows that \(u_n^\prime(\xi) = 0 \text{ for } \xi > 0\). Hence \(u_n(\xi) = u_n = \text{constant for } \xi > 0\). But \(U^\prime(\xi, \zeta) \to 0 \text{ as } \xi \to \infty\). Therefore, \(u_n = 0 \text{ for all } n\), and \(U^\prime = 0 \text{ in } \{(\xi, \zeta) : 0 < \xi, 0 < \zeta < L\}\). Hence \(u^\prime(x, z) = 0 \text{ in } (x_1, x_2) \in \mathbb{R}_0^2 \setminus \Omega\), if \(0 < x_1 < x_1 \tan(\theta_0); 0 < x_2 < x_2^0 \text{ if } x_1 > x_2^0 \cot(\theta_0)\). Owing to the real analyticity of \(u^\prime\) it follows that \(u^\prime = 0 \text{ in } \mathbb{R}_0^2 \setminus \Omega\).

Now we prove that if \(u^\prime = 0 \text{ on } \Gamma_1\), then \(u^\prime = 0 \text{ in } \mathbb{R}_0^2 \setminus \Omega\). But if \(u^\prime = 0 \text{ on } \Gamma_1\), then \(u^\prime = 0 \text{ for } x^2 > x_0^2\). The real analyticity of \(u^\prime\) follows that \(u^\prime = 0 \text{ in } \mathbb{R}_0^2 \setminus \Omega\).

In the following we use \(\Gamma \) to denote either \(\Gamma_1 \) or \(\Gamma_2\).

**Lemma 2.2** Let \(D\) be a bounded domain such that \(\mathbb{R}_0^2 \setminus D\) is connected. Let \(G(\cdot, x^\prime)\) be the outgoing Green's function for \(\mathbb{R}_0^2\) with source \(x^\prime \in \Gamma\).
If \( k^2 \) is not a Dirichlet eigenvalue for the region \( D \), then the restriction of the set \( \{ G(\cdot, x'): x' \in \Gamma \} \) to \( \partial D \) is complete in \( L^2(\partial D) \).

Proof We want to prove that if
\[
\int_{\partial D} \phi(y) G(y, x') \, ds(y) = 0 \quad \text{for all } x' \in \Gamma,
\]
then \( \phi = 0 \) on \( \partial D \).

Let \( \phi \in L^2(\partial D) \) satisfy (2.7). Then the single layer potential
\[
u(x) = \int_{\partial D} \phi(y) G(x, y) \, ds(y) \quad \text{for } x \in \mathbb{R}^3 \setminus \bar{D}
\]
is a solution of Helmholtz equation in \( \mathbb{R}^3 \setminus \bar{D} \) satisfying (1.7)-(1.9), and vanishing on \( \Gamma \). Hence by Lemma 2.1, \( \nu \equiv 0 \) in \( \mathbb{R}^3 \setminus \bar{D} \).

Using the continuity of the single layer potential across the boundary \( \partial D \) and \( k^2 \) being not an eigenvalue it then follows that \( \nu \equiv 0 \) in \( D \). However, due to the jump of the single layer potential across \( \partial D \) it then follows that
\[
\phi = \frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} = 0 \quad \text{on } \partial D,
\]
where \( \partial u^+/\partial n \) and \( \partial u^-/\partial n \) denote the limit of \( \partial u/\partial n \) approaching the boundary from outside and inside of \( D \), respectively.

Theorem 2.3 If \( \Omega_1 \) and \( \Omega_2 \) are two scatterers such that for the fixed wave number \( k^2 \) the scattered waves \( u_1(\cdot, x') \) and \( u_2(\cdot, x') \), corresponding to the scatterers \( \Omega_1 \) and \( \Omega_2 \) respectively, coincide on \( \Gamma \) for all \( x' \in \Gamma \). Then
\[
\Omega_1 = \Omega_2.
\]

Proof Let \( \Omega_1 \) and \( \Omega_2 \) be the two scatterers. Then \( u_1 \) and \( u_2 \) are uniquely determined in \( \mathbb{R}^3 \setminus \Omega_1 \) and \( \mathbb{R}^3 \setminus \Omega_2 \), respectively.

Let \( E \) be the unbounded component of \( \mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2) \). Then for any \( x' \in \Gamma \), \( u'(\cdot, x') := u_1(\cdot, x') - u_2(\cdot, x') \) satisfies
\[
\Delta u' + k^2 u' = 0, \quad \text{in } E,
\]
\[
u' = 0 \quad \text{for } x_3 = 0,
\]
\[
u' = 0 \quad \text{on } \Gamma.
\]
and \( u' \) satisfies the outgoing radiation condition.

Hence, by Lemma 2.1, \( u'(x', x') = 0 \) in \( E \) for any \( x' \in \Gamma \).

Choose \( x_0 \in E \) and consider

\[
\begin{align*}
\Delta w_j + k^2 w_j &= 0, & \text{in } \mathbb{R}^3_{\infty} \setminus \bar{\Omega}_j, & j = 1, 2, \\
Bw_j + BG(\cdot, x_0) &= 0, & \text{on } \partial \Omega_j, & j = 1, 2.
\end{align*}
\]

(2.26) (2.27)

We will show that \( w_1' = w_2' \) in \( E \).

Choose a bounded domain \( B \) such that \( \mathbb{R}^3_{\infty} \setminus B \) is connected, \( \bar{\Omega}_1 \cup \bar{\Omega}_2 \subseteq B \), \( x_0 \notin B \), and \( k^2 \) is not an eigenvalue (i.e. neither a Dirichlet, Neumann nor Impedance eigenvalue) for \( B \). Then by Lemma 2.2, there exists a sequence \( \{v_n\} \) in \( \text{Span}\{G(\cdot, x'): x' \in \Gamma\} \) such that

\[
\|v_n - G(\cdot, x_0)\|_{L^2(\partial B)} \to 0, \text{ as } n \to \infty,
\]

which follows that \( v_n \to G(\cdot, x_0) \) as \( n \to \infty \) uniformly on \( \bar{\Omega}_1 \cup \bar{\Omega}_2 \).

Since the \( v_n \) are linear combinations of Green’s functions (the incident waves), the corresponding scattered waves \( v_{n,1}' \) and \( v_{n,2}' \) for the obstacles \( \Omega_1 \) and \( \Omega_2 \) coincide in \( E \). By the well-posedness of the exterior problem we can conclude that \( v_{n,j}' \to w_j' \) as \( n \to \infty \) uniformly on compact subsets of \( \mathbb{R}^3_{\infty} \setminus \bar{\Omega}_j \) for \( j = 1, 2 \). Hence, \( w_1' = w_2' \) in \( E \).

Now assume that \( \Omega_1 \neq \Omega_2 \). Then without loss of generality, there exists \( x^* \in \partial E \) such that \( x^* \in \partial \Omega_1 \) and \( x^* \notin \Omega_2 \). We choose \( \delta > 0 \) such that the sequence

\[
x_n := x^* + \frac{\delta}{n} \nu(x^*), \quad n = 1, 2, \ldots
\]

is contained in \( E \), where \( \nu(x^*) \) is the outward normal vector at \( x^* \in \partial \Omega_1 \). Consider the solutions \( w_{n,j}' \) to the exterior problem (2.15)(2.16) with \( x_0 \) replaced by \( x_n \). Then \( w_{n,1}' = w_{n,2}' \) in \( E \).

But considering \( w_n' = w_{n,2}' \) as the scattered wave corresponding to the obstacle \( \Omega_1 \), we have that \( w_n' \) is uniformly bounded with respect to the usual Sobolev \( H^1 \) norm on closed subsets of \( \mathbb{R}^3_{\infty} \setminus \bar{\Omega}_2 \), in particular \( a \partial w_n'(x^*)/\partial v + b w_n'(x^*) \) remains bounded as \( n \to \infty \) for any numbers \( a \) and \( b \).

On the other hand, considering \( w_n' = w_{n,1}' \) as the scattered wave corresponding to the obstacle \( \Omega_1 \), we have that
\[ Bw_n(x^*) + B G(x^*, x^n) = 0 \quad \text{for} \quad n = 1, 2, \ldots, \infty. \]

Hence
\[ Bw_n(x^*) = -B G(x^*, x^n) = -a \frac{\partial G(x^*, x^n)}{\partial v} + b G(x^*, x^n) \to \infty, \quad \text{as} \quad n \to \infty, \]
where \( a = 0 \quad \text{and} \quad b = 1 \) for Dirichlet condition, \( a = 1 \quad \text{and} \quad b = 0 \) for Neumann condition, and so on. This is a contradiction. Therefore, \( \Omega_1 = \Omega_2. \)

### 3 A DUAL SPACE INDICATOR METHOD FOR IMAGING AN OBSTACLE IN A SEA WEDGE

Define a scattering operator
\[ U g(x) = \int_{\Gamma} u'(x, x') g(x') \, dx', \quad x \in \Gamma. \quad (3.1) \]

Denote
\[ L^{2,\alpha}(\Gamma) = \{ f : (1 + |x|^2)^{\alpha/2} f \in L^2(\Gamma) \}. \]

**Lemma 3.1** The scattering operator \( hUg \) corresponding to a scatterer \( \Omega \) is an injection from \( L^{2,\alpha}(\Gamma) \) to \( L^{2,\alpha}(\Gamma') \) for \( \alpha > 1/2 \), if \( k^2 \) is neither a Dirichlet, Neumann nor Impedance eigenvalue for \( \Omega \).

**Proof** Assume that \( hUg_1 = hUg_2 \) on \( \Gamma \). Let \( g = g_1 - g_2. \) Then
\[ \int_{\Gamma} u'(x, x') g(x') \, dx' = 0, \quad x \in \Gamma. \quad (3.2) \]

Therefore, the function defined by
\[ w'(x) := \int_{\Gamma} u'(x, x') g(x') \, dx', \quad x \in \mathbb{R}^2_0 \setminus \bar{\Omega} \quad (3.3) \]
satisfies
\[ \Delta w' + k^2 w' = 0, \quad \text{for} \quad x \in \mathbb{R}^2_0 \setminus \bar{\Omega}. \quad (3.4) \]

Moreover,
\[ w' = 0, \quad \text{when} \quad x_2 = 0; \quad \text{and} \quad w' = 0, \quad \text{for} \quad x \in \Gamma, \quad (3.5) \]
and, in addition, \( w' \) satisfies the outgoing radiation condition as \( |x_1| \to \infty \). This implies \( w'(x) \equiv 0 \) in \( \mathbb{R}^2 \setminus \Omega \). Moreover, \( w' \) is the scattered wave corresponding to the incident wave

\[
   w'(x) = \int_\gamma u'(x, x')g(x')\,dx', \quad x \in \mathbb{R}^2 \setminus \Omega. \tag{3.6}
\]

That is,

\[
   B(u' + w') = \int_\gamma (u' + w')(x, x')g(x')\,dx' = 0 \quad \text{on } \partial \Omega. \tag{3.7}
\]

Since \( w' \equiv 0 \) in \( \mathbb{R}^2 \setminus \Omega \), we have

\[
   Bw' = 0 \quad \text{on } \partial \Omega. \tag{3.8}
\]

Since \( k^2 \) is neither a Dirichlet, Neumann nor Impedance eigenvalue, (2.9) implies further that \( w' \equiv 0 \) in \( \Omega \). It implies that \( w' \equiv 0 \) in \( \mathbb{R}^2 \setminus \Omega \). From (13) and that \( u' \) is a Green's function, we know that \( w'(x) \) is a single-layer-potential with density function \( g(x') \). Hence \( g(x') = 0 \) for \( x' \in \Gamma \).

**Lemma 3.2** Let \( bfU' \) be the scattering operator corresponding to a scatterer \( \Omega \), and \( G(x, y) \) be the Green's function for the sea wedge without any obstacle. Then \( G(\cdot, y) \notin \text{Range}(bfU') \) if \( y \notin \Omega \).

**Proof** Otherwise, there were \( y \in \Omega \) and \( g(\cdot, y) \in L^2(\Gamma) \), \( \alpha > 1/2 \), such that

\[
   bfU'g(x, y) = \int_\gamma u'(x, x')g(x', y)\,dx' = G(x, y), \quad \text{for } x \in \Gamma. \tag{3.9}
\]

Then the function

\[
   v(x; y) = \int_\gamma u'(x, x')g(x', y)\,dx', \quad x \in \mathbb{R}^2 \setminus \Omega \tag{3.10}
\]

satisfies the Helmholtz equation in \( \mathbb{R}^2 \setminus \Omega \), and satisfies the outgoing radiation condition. Moreover, \( v(x; y) = G(x, y) \) on \( \Gamma \). Therefore, \( v(x; y) = G(x, y) \) on \( \{x \in \mathbb{R}^2 \setminus \Omega : 0 \leq x_2 \leq x_2'\} \). By continuation,

\[
   \mathbb{R}^2_{\pi/4} := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq x_1 \tan(\pi/4), \ 0 \leq x_1 \leq \infty\},
\]

or in cylindrical coordinates as

\[
   \{(r, \theta) : 0 \leq r < \infty, \ 0 \leq \theta \leq \pi/4\};
\]

where \( \pi/4 \) is the angle of the seabed, \( x_2 = 0 \) is corresponding to the water surface. The obstacle is the interior of the curve \( x = 20 + r \cos(\theta), \ y = 6 + r \sin(\theta) \), where

\[
   r = 1 + 0.9 \sin(\theta), \ 0 \leq \theta \leq 2\pi. \tag{4.1}
\]

and \( \gamma = N \). \( y \) is a fixed point in the searching area \( D \cap \Omega \).
The input data for inverse problems are obtained by solving the direct scattering problem using a boundary integral equation method. When solving the direct scattering problem, we assume the obstacle is soft, i.e., the Dirichlet boundary condition is satisfied. (When solving the inverse problem, we do not use this information.)

We assume that $\Gamma$ and $\Gamma_f$ are the same line. The data $u(x_i; x_j)$ are given at $(x_i; x_j) \in \Gamma_N \times \Gamma_N$ where $\Gamma_N = \{x_n \in \Gamma | n = 0, \ldots, N\}$ for a positive integer $N$. Therefore, we have an $N+1$ by $N+1$ array of data. We approximate the integral by trapezoidal rule. Let $g_j = g(x_j)$, $f_i = G(x_i; y)$, and $S = (s_{ij})_{(N+1) \times (N+1)}$, where $s_{ij} = u(x_i; x_j) \Delta$, $\Delta = (x_N - x_0)/N$ for $0 < i < N$ and $\Delta = (x_N - x_0)/(2N)$ for $i = 0$ and $i = N$. $y$ is a fixed point in the searching area $\Omega$. Let $f = (f_0, \ldots, f_N)^T$ and $g = (g_0, \ldots, g_N)^T$. The integral Eq. (3.1) is approximated by

$$Sg = f.$$ 

Note that $f$ is a vector function of $y \in D$. Hence $g$ is also a vector function of $y$. The matrix $S$ is an ill-conditioned matrix. We use Tikhonov regularization, i.e., we solve the following regularized system instead:

$$(\epsilon I + S^*S)g = S^*f,$$ 

where $S^*$ denotes the conjugate of $S$ and $\epsilon$ is a small parameter. After solving $g$ we compute its $L_2$ norm $\|g(\cdot, y)\|_{L_2(\Gamma)}$ for each $y \in D$ and plot its contour on $D$.

**Example 1** The data are given on $\Gamma = \{(25, 0.1 + (31/15)n) : n = 0, \ldots, 120\}$. (See Fig. 2.) The searching area is the square $D = (17, 23) \times (3, 9)$. The regularization parameter $\epsilon = 10^{-12}$. The contour of $\|g(\cdot, y)\|_{L_2(\Gamma)}, y \in D$ (without filtering) is plotted in Fig. 3(a). In Fig. 3(b) we set $\|g\| = 0$ when it is less than $-0.5$.

**Example 2** The data are given on $\Gamma = \{(5 + (1/4)n, 2) : n = 0, \ldots, 120\}$. (See Fig. 4.) The searching area is the square $D = (17, 23) \times (3, 9)$. The regularization parameter $\epsilon = 10^{-12}$. The contour of $\|g(\cdot, y)\|_{L_2(\Gamma)}, y \in D$ (without filtering) is plotted in Fig. 5(a). In Fig. 5(b) we set $\|g\| = 0$ when it is less than 0.5.
FIGURE 2 Numerical Example 1.

(a) No filtering  (b) Filtering at level -0.50

FIGURE 3 Numerical Example 1.

FIGURE 4 Numerical Example 2.
UNKOWN OBJECT PROBLEM

FIGURE 5 Numerical Example 2.

(a) No filtering  (b) Filtering at level 0.50

References