Transfer functions for a one-dimensional fluid–poroelastic system subject to an ultrasonic pulse

James L. Buchanan a, Robert P. Gilbert b,*, Miao-jung Ou b

a Mathematics Department, United States Naval Academy, Annapolis, MD 21402, USA
b Department of Mathematical Sciences, University of Delaware, Newark, DE 19711, USA

ARTICLE INFO
Article history:
Received 26 January 2011
Accepted 3 February 2011

Keywords:
Poroelastic materials
Laplace transforms
Biot’s equations
Numerical methods
Fast Fourier transforms

ABSTRACT
A one-dimensional model of an in vitro experiment, in which a specimen of cancellous bone is immersed in water and insonified by an ultrasonic pulse, is considered. The modification of the poroelastic model of Biot due to Johnson et al. [D.L. Johnson, J. Koplik, R. Dashen, Theory of dynamic permeability and tortuosity in fluid-saturated porous media, J. Fluid Mech. 176 (1987) 379–402] is used for the cancellous bone segment. By working with series expansions of the Laplace transform in terms of travel-time exponentials, a series of transfer functions for the reflection and transmission of fast and slow waves at the fluid–poroelastic interfaces are derived. The approach obviates numerical solution beyond the discretization involved in the use of the fast Fourier transform.

© 2011 Published by Elsevier Ltd

doi:10.1016/j.nonrwa.2011.02.001

1. Introduction

Cancellous bone consists of a trabecular frame, the interstices of which are filled with blood and fatty marrow. Several authors [1–5] have attempted to model cancellous bone using Biot’s model [6–10] for a poroelastic medium. The survey article of Haire and Langton [11] reviews the successes and failures of early attempts to apply the Biot theory to ultrasound propagation through bone. The more recent application of the modification of Biot’s model due to Johnson et al. [12] by Fellah et al. [4] and Sebaa et al. [5] have produced good agreement with measured results in the in vitro experiment in which a cancellous bone specimen from which the interstitial fatty marrow has been removed is immersed in water and insonified by an ultrasonic pulse.

In this article, it is shown that a simple idea in conjunction with the power of current computer algebra systems can be used to explore the details of transmission and reflection in the in vitro experiment. Reflection and transmission coefficients for the Biot model at a fluid–poroelastic interface for a monochromatic time-harmonic sound source were originally worked out by Stoll and Kan [13]. The case of depth-varying parameters was considered by Stern et al. [14]. Wu et al. [15] developed formulas for determining the amounts of energy transferred to the three types of waves arising in a poroelastic medium. Transfer functions for the polychromatic case are derived in [4]. The approach taken in this article will provide some additional capabilities, particularly, that of differentiating among waves with similar arrival times, but different histories of reflection and transmission.

2. Modeling the in vitro experiment

Consider a poroelastic segment \( \mathcal{B} \) occupying \((0, L)\) with fluids \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) occupying \((-\infty, 0) \) and \((L, \infty) \) respectively with a source at \( x = x_s < 0 \). When the parameters of \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are the same, this is the model used by Fellah et al. [4]
to simulate an experiment in which a cancellous bone specimen is immersed in water and sonified by an ultrasonic pulse.

For the poroelastic medium the modification of Biot’s model due to Johnson et al. [12] will be used. In this model, the dynamic tortuosity $\alpha(\omega)$ is expressed as a function of the asymptotic tortuosity $\alpha_\infty$, pore fluid viscosity $\eta$, pore fluid density $\rho_f$, permeability $k$, porosity $\beta$, the angular frequency $\omega$ and the viscous characteristic length $\Lambda$

$$\alpha(\omega) = \alpha_\infty \left( 1 + \frac{\eta \beta}{\omega \alpha_\infty \rho_f k} \sqrt{1 + \frac{4 \alpha_\infty^2 k^2 \rho_f \omega}{\eta \Lambda^2 \beta^2}} \right) .$$

Let $\hat{g}(s) = L[\{f(t)\}] = \int_0^\infty e^{-st} f(t) \, dt$ denote the Laplace transform of a function $g$. The fluid–bone system is modeled by the transformed equations [4]

$$\frac{\partial^2 \hat{p}_0}{\partial x^2} - \frac{s^2}{c_0^2} \hat{p}_0 = -\hat{f}(s) \delta(x - x_c), \quad -\infty < x < 0$$

$$\begin{bmatrix} P & Q \\ Q & R \end{bmatrix} \begin{bmatrix} \frac{d^2 \hat{u}}{dx^2} \\ \frac{d^2 \hat{U}}{dx^2} \end{bmatrix} = \begin{bmatrix} \hat{\rho}_{11} & \hat{\rho}_{12} \\ \hat{\rho}_{12} & \hat{\rho}_{22} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{U} \end{bmatrix} , \quad 0 < x < L$$

$$\frac{\partial^2 \hat{p}_1}{\partial x^2} - \frac{s^2}{c_1^2} \hat{p}_1 = 0 , \quad L < x < \infty .$$

The quantities $u(x, t)$ and $U(x, t)$ track the motions of the frame and interstitial fluid respectively. For an ultrasonic source $\omega$ is large whence $s \approx i \omega$ and the coefficients in the high-frequency approximation of Johnson et al. [12] become

$$\hat{\rho}_{11} = \rho_{11} + \frac{Z}{\sqrt{s}} , \quad \hat{\rho}_{12} = \rho_{12} - \frac{Z}{\sqrt{s}} , \quad \hat{\rho}_{22} = \rho_{22} + \frac{Z}{\sqrt{s}}$$

(2)

where

$$Z = \frac{2 \beta \alpha_\infty}{A - \sqrt{\rho_f \eta}}$$

and $\rho_{11}, \rho_{12}, \rho_{22}$ are the mass-coupling terms in the Biot’s model defined in terms of the density of the frame material $\rho_s$, the pore fluid density $\rho_f$, $\alpha_\infty$ and $\omega$

$$\rho_{12} := -\beta \rho_f (\alpha_\infty - 1) , \quad \rho_{22} := \beta \rho_f \alpha_\infty$$

$$\rho_{11} := (1 - \beta) \rho_s + \beta \rho_f (\alpha_\infty - 1) .$$

The effective elastic constants $P, Q, and R$ are related to $\beta$, bulk modulus of the pore fluid $K_f$, bulk modulus of the trabecular bone $K_s$, bulk modulus of the porous skeletal frame $K_b$ and the shear modulus of the composite as well as the skeletal frame $G$

$$P := \frac{(1 - \beta) \left( 1 - \beta \frac{K_s}{K_f} \right) K_s + \beta \frac{K_s}{K_f} K_b}{1 - \beta - \frac{K_b}{K_s} + \beta \frac{K_s}{K_f}} + \frac{4}{3} G$$

$$Q := \frac{(1 - \beta - \frac{K_s}{K_f}) \beta K_s}{1 - \beta - \frac{K_b}{K_s} + \beta \frac{K_s}{K_f}}$$

$$R := \frac{\beta \beta K_s}{1 - \beta - \frac{K_b}{K_s} + \beta \frac{K_s}{K_f}} .$$

The system of equations for the poroelastic segment in (1) can be written

$$\begin{bmatrix} \frac{d^2 \hat{u}}{dx^2} \\ \frac{d^2 \hat{U}}{dx^2} \end{bmatrix} = \begin{bmatrix} s^2 \\ \frac{s^2}{PR - Q^2} \end{bmatrix} \begin{bmatrix} R \hat{\rho}_{11} - Q \hat{\rho}_{12} & R \hat{\rho}_{12} - Q \hat{\rho}_{22} \\ -Q \hat{\rho}_{11} + P \hat{\rho}_{12} & -Q \hat{\rho}_{12} + P \hat{\rho}_{22} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{U} \end{bmatrix} .$$

In terms of the eigenvalues $\lambda_j$ and corresponding eigenvectors $v_j, j = 1, 2, \ldots$ of the matrix on the right hand side the system becomes

$$\begin{bmatrix} \frac{d^2 \hat{u}}{dx^2} \\ \frac{d^2 \hat{U}}{dx^2} \end{bmatrix} = \frac{s^2}{PR - Q^2} \lambda_j v_j \left( \hat{u} \right) .$$
where
\[ \mathbf{V} := (\mathbf{v}_1 \quad \mathbf{v}_2) : \quad \mathbf{A} := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \]

With
\[ b := -R\tilde{\rho}_{11} + 2Q\tilde{\rho}_{12} - P\tilde{\rho}_{22}, \]
\[ c := RP(\tilde{\rho}_{11}\tilde{\rho}_{22} - \tilde{\rho}_{12}^2) + Q^2(\tilde{\rho}_{12}^2 - \tilde{\rho}_{22}\tilde{\rho}_{11}) \]
the eigenvalues are
\[ \lambda_1 = \frac{-b - \sqrt{b^2 - 4c}}{2}; \quad \lambda_2 = \frac{-b + \sqrt{b^2 - 4c}}{2}. \]

The solution to (1) then has the form
\begin{align*}
\hat{p}_0 &= C_1 e^{sx/c_0}, \quad -\infty < x < x_4, \\
\hat{p}_0 &= C_2 e^{sx/c_0} + C_3 e^{-sx/c_0}, \quad x_4 < x < 0, \\
\hat{u}_0 &= C_4 v_1 e^{sx/c_0} + C_5 v_1 e^{-sx/c_0} + C_6 v_1 e^{sx/c_0} + C_7 v_3 e^{-sx/c_0}, \quad 0 < x < L, \\
\hat{U}_0 &= C_4 v_2 e^{sx/c_0} + C_5 v_2 e^{-sx/c_0} + C_6 v_2 e^{sx/c_0} + C_7 v_2 e^{-sx/c_0}, \quad 0 < x < L, \\
\hat{p}_1 &= C_8 e^{-sx/c_1}, \quad x > L
\end{align*}

where the speeds of the fast and slow waves are
\[ c_f := \left(\frac{\lambda_1}{PR - Q^2}\right)^{-1/2}, \quad c_s := \left(\frac{\lambda_2}{PR - Q^2}\right)^{-1/2} \]

and the notation \( v_j = v_{1j}, v_0 = v_{2j}, j = 1, 2 \) has been adopted. In the cancellous segment normal frame stress and pore fluid pressure are related to the displacements by
\[ \hat{\sigma}_{xx} = P \frac{\partial\hat{u}_0}{\partial x} + Q \frac{\partial\hat{U}_0}{\partial x}, \quad \hat{\sigma} = Q \frac{\partial\hat{u}_0}{\partial x} + R \frac{\partial\hat{U}_0}{\partial x}. \]

At the interface continuity of fluid movement in and out of the frame, continuity of pore fluid pressure, and continuity of total normal pressure
\begin{align*}
\hat{U}_0(0-, s) &= \beta \hat{U}_0(0+, s) + (1 - \beta)\hat{u}_0(0+, s) \\
\hat{p}_0(0-, s) &= \hat{\sigma}(0+, s)/\beta \\
\hat{p}_0(0-, s) &= \hat{\sigma}_{xx}(0+, s) / \beta \hat{\sigma}(0+, s) \\
\hat{U}_1(L+, s) &= \beta \hat{U}_1(L-, s) + (1 - \beta)\hat{u}_1(L-, s) \\
\hat{p}_1(L+, s) &= \hat{\sigma}(L-, s)/\beta \\
\hat{p}_1(L+, s) &= \hat{\sigma}_{xx}(L-, s) / \beta \hat{\sigma}(L-, s)
\end{align*}

are required. Substituting this into (8) and using the jump conditions for the delta function
\begin{align*}
\hat{p}_0(x_4+) - \hat{p}_0(x_4-) &= 0 \\
\frac{\partial\hat{p}_0}{\partial x}(x_4+) - \frac{\partial\hat{p}_0}{\partial x}(x_4-) &= -\frac{f}{c_f^2}
\end{align*}

results in a linear system of equations for the coefficients \( C_j, j = 1, \ldots, 8 \) which can be solved exactly using a computer algebra system. All results in this article were derived using Maple, Version 11.

3. Numerical simulation of an in vitro experiment for porous bone

In numerical simulations the incident pulse used is
\[ f(t) = \exp(-(t - t_0)^2/\Sigma) \sin \omega_0 t \]
with \( \omega_0 = 2.25 \text{ MHz, } \omega_0 = 2\pi f_0, t_0 = 1/f_0, t_c = 1.75 f_0, \Sigma = t_c^2. \ Fig. 1 \) shows \( f \) and its spectral content. Laplace and inverse Laplace transforms are approximated using the fast Fourier transform as described in Appendix B. Since the high-frequency
approximation (2) is used, inclusion of low frequencies will introduce inaccuracies. For this reason all Laplace transforms are set to zero below 0.5 MHz before inversion. In the Laplace transform approximation (17) a value \( T \) is chosen so that \([0, T]\) contains the effective support of the function \( f \). In (17) a target value of \( \Omega = 5000 \) is used, but is modified slightly so that the number of points \( N \) used is a power of 2. The sets of parameters used in these simulations are those of specimens M1–M3 of Fellah et al. [4], which are designated M1F04–M3F04, and M1–M3 of Sebaa et al. [5], which are designated M1S06–M3S06. The parameters for these specimens are given in Appendix C.

The coefficients \( C_i \) in (3) are functions of the travel-time exponentials

\[
e_1 = \exp(-sL/c_f), \quad e_2 = \exp(-sL/c_i).
\]

The solution \( \hat{p}_1 \) has the form

\[
\hat{p}_1 = C_0 e^{-sx/c_1} = \frac{A_{01} \hat{e}_2 + A_{10} \hat{e}_1 + A_{12} \hat{e}_1 \hat{e}_2^2 + A_{21} \hat{e}_1^2 \hat{e}_2}{B_{00} + B_{02} \hat{e}_2^2 + B_{11} \hat{e}_1 \hat{e}_2 + B_{20} \hat{e}_1^2 + B_{22} \hat{e}_1^2 \hat{e}_2} \exp\left(-\frac{s(x/c_0 + (x-L)/c_1)}{c_0}\right) \hat{f}(s).
\]

Let \( (\hat{p})_{mn} \) denote the coefficient of \( e_n^m \hat{e}_2^m \) of the Taylor series expansion of \( \hat{p} \). The wave \( p_{nm} = L^{-1} \{ (\hat{p})_{mn} e_n^m \} \) is then the sum of all waves arriving at position \( x > L \) after having traversed the cancellous segment \( Bm \) times as a fast wave and \( n \) times as a slow wave. Fig. 2 shows that the wave \( p = L^{-1} \{ \hat{p}_1 \} \) for the specimen M3S06 with transform given in (3) is the sum of the seven arrivals of the form \( p_{mn} \).

4. Transfer coefficients for fluid layers

While the in vitro experiment leads to an \( 8 \times 8 \) linear system that can be solved exactly, the time and memory requirements will rise rapidly with the number of media segments. Thus an approach which obviates the need to solve the large systems associated with many media segments may be of use, especially since the linear systems arising from many segment problems tend to be poorly conditioned numerically whence numerical solution is not an alternative. The approach discussed in Section 3 can be adapted to this end. It will first be illustrated by finding the well-known transmission and reflection coefficients for fluid layers. Pressure \( p_n \) and displacement \( U_n, n = 0, 1, 2 \) in a system of three fluids \( \mathcal{F}_n \) occupying the regions \((-\infty, 0), (0, L), (L, \infty)\) and insonified by a source of magnitude \( f(t) \) located at \( x = x_s < 0 \) is described by the transformed problem

\[
\frac{\partial^2 \hat{p}_0}{\partial x^2} - \frac{s^2}{c_0^2} \hat{p}_0 = \frac{\hat{f}(s)}{c_0} \delta(x - x_s), \quad -\infty < x < 0
\]

\[
\frac{\partial^2 \hat{p}_1}{\partial x^2} - \frac{s^2}{c_1^2} \hat{p}_1 = 0, \quad 0 < x < L
\]

\[
\frac{\partial^2 \hat{p}_2}{\partial x^2} - \frac{s^2}{c_2^2} \hat{p}_2 = 0, \quad x > L
\]

\[
s^2 \rho_0 \hat{U}_0 = -\frac{\partial \hat{p}_0}{\partial x}, \quad s^2 \rho_1 \hat{U}_1 = -\frac{\partial \hat{p}_1}{\partial x}, \quad s^2 \rho_2 \hat{U}_2 = -\frac{\partial \hat{p}_2}{\partial x}
\]
Fig. 2. Top: pressure $p$ calculated from \((3)\) is virtually the same as the sum of the seven arrivals $p_{10}, p_{01}, p_{30}, p_{21}, p_{12}, p_{50}, p_{03}$. Bottom: the seven arrivals plotted individually. The length of the bone segment was $L = 0.5$ cm and the source and receiver were 1 cm to the left and right of the bone segment respectively. Both fluids were assumed to be water with the density and bulk modulus given in Appendix C.

\[
\begin{align*}
\hat{p}_0(0) &= \hat{p}_1(0) \\
\hat{U}_0(0) &= \hat{U}_1(0) \Rightarrow \frac{1}{\rho_0} \frac{\partial \hat{p}_0}{\partial x}(0) = \frac{1}{\rho_1} \frac{\partial \hat{p}_1}{\partial x}(0) \\
\hat{p}_1(L) &= \hat{p}_2(L) \\
\frac{1}{\rho_1} \frac{\partial \hat{p}_1}{\partial x}(L) &= \frac{1}{\rho_2} \frac{\partial \hat{p}_2}{\partial x}(L)
\end{align*}
\]

where $c_n$ and $\rho_n$ are the wave speed and density of segment $n$. The solution has the form

\[
\begin{align*}
\hat{p}_0 &= C_1 e^{sx/c_0}, \quad -\infty < x < x_s \\
\hat{p}_0 &= C_2 e^{-sx/c_0} + C_3 e^{sx/c_0}, \quad x_s < x < 0 \\
\hat{p}_1 &= C_4 e^{-sx/c_1} + C_5 e^{sx/c_1}, \quad 0 < x < L \\
\hat{p}_2 &= C_6 e^{-sx/c_2}, \quad x > L
\end{align*}
\]

Substituting this into \((8)\) and using the jump conditions \((6)\) at the source position $x_s$ results in a linear system of equations for the coefficients $C_j, j = 1, \ldots, 6$ which can be solved exactly using a computer algebra system. The function

\[
\hat{p} = C_2 e^{-sx/c_0} = \frac{1}{2c_0} \frac{\hat{f}(s)}{s} \exp(-s(x - x_s)/c_0)
\]

is the transform of a half-height wave traveling rightward from the source. Let

\[
\mathcal{T}(\Sigma_0 | F_0) = \frac{1}{2c_0}
\]

denote transfer coefficient for transmission from the source into fluid $F_0$. The pressures and displacements are functions of the travel-time exponential $e_1 = \exp(-sL/c_1)$. As above let $(F)_{\alpha}$ denote the coefficient of $e_1^\alpha$ in the Taylor series expansion of a function $F(e_1)$. The function

\[
\hat{p} = (C_4) e^{-sx/c_1} = \frac{c_1\rho_1}{c_0\rho_0 + c_1\rho_1} \frac{\hat{f}(s)}{s} \exp(-s(x_s/c_0 + x/c_1))
\]

is the transform of the direct (unreflected) arrival of the pulse at position $x$ in fluid $F_1$. Letting

\[
\mathcal{T}(F_0 | F_1) = \frac{2c_1\rho_1}{c_0\rho_0 + c_1\rho_1}
\]

gives

\[
\hat{p} = \mathcal{T}(\Sigma_0 | F_0) \mathcal{T}(F_0 | F_1) \frac{\hat{f}(s)}{s} \exp(-s(x_s/c_0 + x/c_1)).
\]
The function
\[ \hat{p} = (C_3)_{0} e^{s(x/c_0)} = \frac{c_1 \rho_1 - c_0 \rho_0}{c_0 \rho_0 + c_1 \rho_1} \frac{f(s)}{2c_0 s} \exp(-s((-x_s - x)/c_0)), \quad x_s < x < 0 \]
represents the transform of the reflection of the source pulse from the interface \( x = 0 \). Setting
\[ \mathcal{R}(\mathcal{F}_0, \mathcal{F}_1) = \frac{c_1 \rho_1 - c_0 \rho_0}{c_0 \rho_0 + c_1 \rho_1} \]
gives
\[ \hat{p} = \mathcal{T}(\Sigma_0|\mathcal{F}_0) \mathcal{R}(\mathcal{F}_0|\mathcal{F}_1) \frac{f(s)}{s} \exp(-s((-x_s - x)/c_0)). \]
All waves should be expressible in the form \( P(s) e^{-s \tau} \) where \( P \) is a product of the transmission and reflection coefficients defined above and \( \tau \) is the time of arrival. For instance a wave arriving at position \( x > L \) after having been reflected once back and forth between \( x = 0 \) and \( x = L \) arrives at time \( \tau = -x_s/c_0 + 3L/c_1 + (x-L)/c_2 \) is given by
\[ \hat{p} = (C_0)_{0} e^{3s/c_2} \]
\[ = 2\left( \frac{c_1 \rho_1 - c_0 \rho_0}{c_0 \rho_0 + c_1 \rho_1} \right) \left( \frac{c_1 \rho_1 - c_2 \rho_0}{c_0 \rho_0 + c_1 \rho_1} \right) \frac{f(s)}{s} \exp(-s((-x_s - x)/c_0 + 3L/c_1 + (x-L)/c_2)) \]
\[ = \mathcal{T}(\Sigma_0|\mathcal{F}_0) \mathcal{T}(\mathcal{F}_0|\mathcal{F}_1) \mathcal{R}(\mathcal{F}_1|\mathcal{F}_2) \mathcal{R}(\mathcal{F}_2|\mathcal{F}_0) \mathcal{T}(\mathcal{F}_1|\mathcal{F}_2) \cdot \frac{f(s)}{s} \exp(-s((-x_s - x)/c_0 + 3L/c_1 + (x-L)/c_2)). \]
That the transmission or reflection of any pulse can be expressed as
\[ \hat{p} = \mathcal{H}(\mathcal{F}_n|\mathcal{F}_{n+1}) \frac{\hat{g}(s)}{s} \exp(-s \tau) \]
where \( \mathcal{H}(\mathcal{F}_n|\mathcal{F}_{n+1}) \) depends only upon the parameters of layers \( \mathcal{F}_n \) and \( \mathcal{F}_{n+1} \) seems reasonable in view of the expectation that what transpires at a particular interface should not depend upon the history of the pulse. Thus it is expected that such factorizations are possible in any layered medium, be the layers fluid, elastic, or poroelastic.

5. Transfer functions at a fluid–poroelastic interface

In view of the interface conditions (5), total pressure
\[ p_b = -(\sigma_{ax} + \sigma) \]
is a natural measure of the pressure within the bone segment. From (3) and (4) \( p_b \) is the sum of the four waves
\[ \hat{p}_{b_4} = -\frac{P v_1 + Q v_2 + Q v_1 + R v_2}{c_f} s c_4 \exp(s x/c_f) \]
\[ \hat{p}_{b_5} = \frac{P v_1 + Q v_2 + Q v_1 + R v_2}{c_f} s c_5 \exp(-s x/c_f) \]
\[ \hat{p}_{b_6} = -\frac{P v_1 + Q v_2 + Q v_1 + R v_2}{c_s} s c_6 \exp(s x/c_s) \]
\[ \hat{p}_{b_7} = \frac{P v_1 + Q v_2 + Q v_1 + R v_2}{c_s} s c_7 \exp(-s x/c_s). \]
The portion of the incident wave that transmits into the cancellous segment as a fast wave arrives at position \( x \) at time \( \tau = -x_s/c_0 + x/c_s \) is thus given by
\[ \hat{p}_{b_5} = \frac{\exp(-s((-x_s/c_0 + x/c_s)) f(s))}{\Phi(c_0, \rho_0, c_f, v_{11}, v_{12}, c_s, v_{11}, v_{12})} \]
where \( \Psi, \Phi, \) and all subsequent constituent functions of the transfer functions are given in Appendix A. This gives the formula for \( \mathcal{T}(\mathcal{F}_0|\mathcal{F}_2) \) in Table 1. The arrival time for the wave that traverses the cancellous segment fast wave and reflects back as a slow wave arrives at position \( x \) at time \( \tau = -x_s/c_0 + L/c_f(s) + (L-x)/c_f(s) \). It is given by (cf (7))
\[ \hat{p}_{b_6} = -\frac{\exp(-s((-x_s/c_0 + x/c_s)) f(s))}{\Phi(c_0, \rho_0, c_f, v_{11}, v_{12}, c_s, v_{11}, v_{12})} \]
\[ \cdot \frac{\Psi(v_{11}, v_{12}, v_{11}, v_{12})}{\Phi(c_0, \rho_0, c_f, v_{11}, v_{12}, c_s, v_{11}, v_{12})} \frac{\exp(-s x/c_f)}{c_f} \]
\[ = \mathcal{T}(\Sigma_0|\mathcal{F}_0) \mathcal{T}(\mathcal{F}_0|\mathcal{F}_2) \frac{\Psi(v_{11}, v_{12}, v_{11}, v_{12})}{\Phi(c_0, \rho_0, c_f, v_{11}, v_{12}, c_s, v_{11}, v_{12})} \frac{\exp(-s x/c_f)}{c_f} \]
which gives the entry for $\mathcal{R}(\mathcal{B}|\mathcal{F}_1 \rightarrow S)$ in Table 1. The remaining entries in the table arise from similar considerations. All transfer functions conform to the expectation that they depend only upon the parameters of the media on each side of the interface at $x = 0$.

With the transfer functions defined in Table 1 the pressures shown in Fig. 2 can be computed as

$$
\begin{align*}
\hat{p}_{10} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, F)\mathcal{T}(\mathcal{B}|\mathcal{F}_1, F) \\
\hat{p}_{01} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, S)\mathcal{T}(\mathcal{B}|\mathcal{F}_1, S) \\
\hat{p}_{30} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, F)\mathcal{R}(\mathcal{B}|\mathcal{F}_1 F \rightarrow F)\mathcal{R}(\mathcal{B}|\mathcal{F}_0 F \rightarrow F) \cdot \mathcal{T}(\mathcal{B}|\mathcal{F}_1, F) \\
\hat{p}_{03} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, S)\mathcal{R}(\mathcal{B}|\mathcal{F}_1 S \rightarrow S)\mathcal{R}(\mathcal{B}|\mathcal{F}_0 S \rightarrow S) \cdot \mathcal{T}(\mathcal{B}|\mathcal{F}_1, S) \\
\hat{p}_{21} &= p_{\text{FFS}} + p_{\text{SSF}} + p_{\text{SSF}} \\
\hat{p}_{12} &= p_{\text{SSF}} + p_{\text{SSS}} + p_{\text{SSF}} \\
\hat{p}_{50} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, F)\mathcal{R}(\mathcal{B}|\mathcal{F}_1 F \rightarrow F)^2\mathcal{R}(\mathcal{B}|\mathcal{F}_0 F \rightarrow F)^2 \cdot \mathcal{T}(\mathcal{B}|\mathcal{F}_1, F)
\end{align*}
$$

where

$$
\begin{align*}
\hat{p}_{\text{FFS}} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, F)\mathcal{R}(\mathcal{B}|\mathcal{F}_1 F \rightarrow F)\mathcal{R}(\mathcal{B}|\mathcal{F}_0 F \rightarrow S)\mathcal{T}(\mathcal{B}|\mathcal{F}_1, S) \\
\hat{p}_{\text{SSF}} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, F)\mathcal{R}(\mathcal{B}|\mathcal{F}_1 F \rightarrow S)\mathcal{R}(\mathcal{B}|\mathcal{F}_0 S \rightarrow F)\mathcal{T}(\mathcal{B}|\mathcal{F}_1, F) \\
\hat{p}_{\text{SSF}} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, S)\mathcal{R}(\mathcal{B}|\mathcal{F}_1 S \rightarrow F)\mathcal{R}(\mathcal{B}|\mathcal{F}_0 S \rightarrow F)\mathcal{T}(\mathcal{B}|\mathcal{F}_1, F) \\
\hat{p}_{\text{SSF}} &= \mathcal{T}(\Sigma_0|\mathcal{F}_0)\mathcal{T}(\mathcal{F}_0|\mathcal{B}, S)\mathcal{R}(\mathcal{B}|\mathcal{F}_1 S \rightarrow S)\mathcal{R}(\mathcal{B}|\mathcal{F}_0 S \rightarrow S)\mathcal{T}(\mathcal{B}|\mathcal{F}_1, S).
\end{align*}
$$

The combined time required to compute the seven constituent pressures in (11) on a laptop PC with 2 GHz CPU running MATLAB was about one-fifth of a second. The number of points used in the fast Fourier transforms was 4096. Fig. 3 shows $p_{21}$ and $p_{12}$ and their three constituents. Figs. 4 and 5 compare the transmission and reflection transfer functions of cancellous bone from and to a fluid with those of a fluid with similar density, $\rho = \tilde{\rho}_1(s)$ or $\tilde{\rho}_2(s)$, and wave speed $c = c_1(s)$ or $c_2(s)$. The fast wave transfer functions differ more from their fluid counterparts than the slow wave transfer functions.

6. Apportionment of energy

In the fluid to the left and right of the bone the flux of energy across an interface at position $x$ is given by

$$
EF = \int_0^\infty p(x, t) \frac{\partial U}{\partial t}(x, t) \, dt.
$$
Fig. 3. Top: the composite waves $p_{21}$ and $p_{12}$ shown in Fig. 2. Bottom: the three constituents $p_{FFS}$, $p_{FSF}$, $p_{SFF}$ of $p_{21}$, and $p_{SSF}$, $p_{SFS}$, $p_{FSS}$ of $p_{12}$. The waves $p_{FFS}$ and $p_{SFF}$ are identical, as are $p_{SSF}$ and $p_{SFS}$.

Fig. 4. Magnitude of transfer functions for a poroelastic medium compared to those of a fluid with similar wavespeed and density. Top left: solid line: $\mathcal{T}(F_0|B,F)$; dashed line: $\frac{2\rho_1}{c_0 \rho_1 + s \rho_1}$. Top right: solid line: $\mathcal{T}(F_0|B,S)$; dashed line: $\frac{2\rho_0}{c_0 \rho_0 + s \rho_0}$. Bottom left: solid line: $\mathcal{T}(B|F_1,F)$; dashed line: $\frac{2\rho_1}{c_0 \rho_1 + s \rho_1}$. Bottom right: solid line: $\mathcal{T}(B|F_1,S)$; dashed line: $\frac{2\rho_0}{c_0 \rho_0 + s \rho_0}$. Circles mark the center frequency of the pulse.

For a wave $p_1$ with transform $\tilde{p}_1 = C_8 e^{-sx/c_1}$ traveling to the right of the bone (cf. (3), (8))

$$
L \left\{ \frac{\partial U}{\partial t} \right\} = s \hat{U}_1 = -\frac{1}{s \rho_1} \frac{\partial \hat{p}_1}{\partial x} = -\frac{1}{s \rho_1} C_8 e^{-sx/c_1} \left( -\frac{s}{c_1} \right) = \frac{1}{\rho_1 c_1} \hat{p}_1
$$
Fig. 5. Magnitude of transfer functions for a poroelastic medium compared to those of a fluid with similar wave speed and density. Top left: solid line: $R(\mathcal{B}|F \rightarrow F)$; dashed line: $\frac{1}{2} \frac{\partial \sigma_{xx}}{\partial t} + \rho c f \frac{\partial \rho}{\partial t}$; top right: solid line: $R(\mathcal{B}|S \rightarrow S)$; dashed line: $\frac{1}{2} \frac{\partial \sigma_{zz}}{\partial t} + \rho \frac{\partial \rho}{\partial t}$; bottom left: solid line: $R(\mathcal{B}|F \rightarrow S)$; bottom right: solid line: $R(\mathcal{B}|S \rightarrow F)$; circles mark the center frequency of the pulse.

and thus the standard formula

$$EF = \frac{1}{\rho_1 c_1} \int_0^\infty p_1(x, t)^2 dt$$

is obtained.

In the bone segment the energy flux is

$$EF = -\int_0^\infty \left( \sigma_{xx}(x, t) \frac{\partial}{\partial t} u_b(x, t) + \sigma(x, t) \frac{\partial}{\partial t} U_b(x, t) \right) dt.$$ (12)

From (4)

$$\frac{\partial \hat{u}_b}{\partial x} = -\frac{Q \hat{\sigma} - R \hat{\sigma}_{xx}}{P R - Q^2} \quad \frac{\partial \hat{U}_b}{\partial x} = \frac{P \hat{\sigma} - Q \hat{\sigma}_{xx}}{P R - Q^2}$$ (13)

and thus for the rightward traveling waves

$$\hat{u}_{bf} = C_5 v_{f1} \exp(-sx/c_f), \quad \hat{U}_{bf} = C_5 v_{f2} \exp(-sx/c_f),$$ (14)

(3) and (4) give

$$\hat{\sigma}_{xxf} = (P v_{f1} + Q v_{f2}) C_5 \exp(-sx/c_f) \left( -\frac{s}{c_f} \right)$$

$$= -\frac{P v_{f1} + Q v_{f2}}{P v_{f1} + Q v_{f2} + Q v_{f1} + R v_{f2}} \hat{p}_{bs}$$

$$\hat{\sigma}_f = (Q v_{f1} + R v_{f2}) C_5 \exp(-sx/c_f) \left( -\frac{s}{c_f} \right)$$

$$= -\frac{Q v_{f1} + R v_{f2}}{P v_{f1} + Q v_{f2} + Q v_{f1} + R v_{f2}} \hat{p}_{bs}.$$
Fig. 6. Top: fast and slow wave portions of the pressure wave $-(\sigma_{xx} + \sigma)$ passing positions $x = 0.1L$ and $0.9L$ for $L = 5$ mm. Bottom: the four waves reflected from the right edge of the bone specimen when passing the midpoint of the bone segment. The specimen was M3S06.

From (13) and (14)

$$-s\frac{\partial \tilde{u}_b}{\partial t} = \frac{Q\tilde{\sigma} - R\tilde{\sigma}_{xx}}{PR - Q^2}$$

$$-\frac{s}{c_f} \frac{\partial \tilde{u}_b}{\partial t} = \frac{P\tilde{\sigma} - Q\tilde{\sigma}_{xx}}{PR - Q^2}$$

and thus the functions occurring in (12) are given by

$$\sigma_{xxf} = -L^{-1} \left\{ \frac{Pv_f + Qv_f}{Pv_f + Qv_f + Qv_f + Pv_f} P\tilde{s} - R\tilde{s}_{xx} \right\}$$

$$\sigma_f = -L^{-1} \left\{ \frac{Qv_f}{Qv_f + Rv_f} P\tilde{s} - Q\tilde{s}_{xx} \right\}$$

$$\frac{\partial u_{bf}}{\partial t} = L^{-1} \left\{ \frac{Q\tilde{s}_f - R\tilde{s}_{xxf}}{PR - Q^2} \right\} , \quad \frac{\partial U_{bf}}{\partial t} = -L^{-1} \left\{ \frac{P\tilde{s}_f - Q\tilde{s}_{xxf}}{PR - Q^2} \right\} .$$

With the corresponding terms for the slow wave, the energy flux across $x = 0$ due to the incident wave is

$$EF = -\int_0^\infty \left[ \left( \sigma_{sxx}(0, t) + \sigma_{sxs}(0, t) \right) \frac{\partial}{\partial t} \left( u_{bf}(0, t) + u_{bs}(0, t) \right) + \left( \sigma_f(0, t) + \sigma_s(0, t) \right) \frac{\partial}{\partial t} \left( U_{bf}(0, t) + U_{bs}(0, t) \right) \right] dt$$

$$= -\int_0^\infty \left[ \sigma_{sxs}(0, t) \frac{\partial}{\partial t} u_{bf}(0, t) + \sigma_{sxx}(0, t) \frac{\partial}{\partial t} u_{bs}(0, t) + \sigma_f(0, t) \frac{\partial}{\partial t} U_{bf}(0, t) + \sigma_s(0, t) \frac{\partial}{\partial t} U_{bs}(0, t) \right] dt$$

$$\times \left[ -\int_0^\infty \left[ \sigma_{sxs}(0, t) \frac{\partial}{\partial t} u_{bf}(0, t) + \sigma_{sxx}(0, t) \frac{\partial}{\partial t} u_{bs}(0, t) + \sigma_f(0, t) \frac{\partial}{\partial t} U_{bf}(0, t) + \sigma_s(0, t) \frac{\partial}{\partial t} U_{bs}(0, t) \right] dt \right] .$$

\[ (15) \]

Table 2 shows the apportionment of the energy of the incident wave (9) at the left fluid–bone interface $x = 0$. Only a small percentage of the energy is reflected back into fluid $F_b$. The negative flux $EF_{fs}(0)$ arising from the multiplicative cross-terms in (15) represents a correction of at most a few percent and thus the “pure” fast and slow wave fluxes $E_f(0)$ and $E_s(0)$ give a good approximation to the apportionment of energy between the two wave types.

In the presence of a viscous pore fluid, wave traveling within the bone will be subject to attenuation. Fig. 6 shows the pressure waves of the form $p_b = -(\sigma_{xx} + \sigma)$ as they pass various positions within the bone.
Table 2
Portions of the incident wave energy which are reflected back to the fluid (Φ(F₀ | B)), transmitted into the bone as a fast wave (Ψ(F₀ | B, F)), and transmitted into the bone as a slow wave (Ψ(F₀ | B, S)) at the left fluid–bone interface. The length of the bone segment was 5 mm.

|       | EF₁(F₀, F₀ | B) (%) | EF₂(F₀, F₀ | B) (%) | EF₃(F₀, F₀ | B) (%) | EF₄(F₀, F₀ | B) (%) |
|-------|------------|------------|------------|------------|
| M1F04 | 0.86       | 8.90       | -0.90      | 91.15      |
| M2F04 | 1.44       | 14.99      | -2.57      | 86.14      |
| M3F04 | 0.36       | 19.96      | -5.71      | 85.39      |
| M1S06 | 2.55       | 18.09      | -0.34      | 79.70      |
| M2S06 | 0.86       | 17.21      | -0.71      | 82.65      |
| M3S06 | 2.54       | 38.86      | -0.12      | 58.72      |

Table 3
Approximate percentages of the energy of the incident pulse apportioned to the four types of reflections that occur at the right edge of the bone. Fluxes due to multiplicate cross-terms are neglected. The length of the bone segment was 5 mm. I: EF₁(F₀ | F₀ | F₀) (I); II: EF₂(F₀ | F₀ | F₀) (II); III: EF₃(F₀ | F₀ | F₀) (III); IV: EF₄(F₀ | F₀ | F₀) (IV).

<table>
<thead>
<tr>
<th></th>
<th>I (%)</th>
<th>II (%)</th>
<th>III (%)</th>
<th>IV (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1F04</td>
<td>1.08</td>
<td>0.05</td>
<td>0.31</td>
<td>0.07</td>
</tr>
<tr>
<td>M2F04</td>
<td>0.85</td>
<td>0.04</td>
<td>0.03</td>
<td>0.01</td>
</tr>
<tr>
<td>M3F04</td>
<td>0.32</td>
<td>0.05</td>
<td>0.09</td>
<td>0.03</td>
</tr>
<tr>
<td>M1S06</td>
<td>6.38</td>
<td>0.33</td>
<td>3.21</td>
<td>1.03</td>
</tr>
<tr>
<td>M2S06</td>
<td>3.65</td>
<td>0.34</td>
<td>1.99</td>
<td>0.58</td>
</tr>
<tr>
<td>M3S06</td>
<td>13.25</td>
<td>0.75</td>
<td>11.54</td>
<td>2.44</td>
</tr>
</tbody>
</table>

Table 4
Energy proportions of the fast and slow waves that are transmitted through the bone without reflection. The length of the bone segment was 5 mm.

|       | EF₁(F₀ | F₀ | F₀) (%) | EF₂(F₀ | F₀ | F₀) (%) |
|-------|-------|--------|---------|
| M1F04 | 0.11  | 2.21   |
| M2F04 | 0.17  | 0.11   |
| M3F04 | 0.09  | 0.35   |
| M1S06 | 1.61  | 8.21   |
| M2S06 | 0.88  | 7.63   |
| M3S06 | 11.62 | 5.86   |

Tables 3 and 4 indicate the approximate percentages of the energy of the incident pulse apportioned to various transmitted and reflected waves.

7. Conclusion

The approach described above has two advantages. First, it permits assessment of the relative magnitudes of the various wave types and their transmissions and reflections. Second, it obviates the need of numerical solution beyond the use of the fast Fourier transform.

Appendix A. Constituent functions of the transfer functions

\[ \Phi(c, \rho, c₁, v₁, c₂, v₁, v₂) = -\beta Q v₂ v₁ c₂ c₁ \rho + \beta² Q v₂ v₁ c₂ c₁ \rho + v₂ β² c₂ c₁ R v₂ v₂ \rho + \beta² v₂ v₁ v₁ c₂ c₁ \rho + \beta v₁ v₂ c₂ c₁ \rho - Q v₁ v₂ c₂ c₁ \rho + \beta Q v₁ v₂ c₂ c₁ \rho - \beta² v₂ v₂ c₂ c₁ \rho + Q v₁ v₂ c₂ c₁ \rho \]

\[ \Psi(v₁, v₂, c₁, v₁, v₂) = (P v₁ + Q v₂ + Q v₁ + R v₁)(\beta R v₂ - v₁ Q + \beta P v₁ - R v₂ + \beta v₂ Q + v₂ Q) \]

\[ \Upsilon(c, \rho, c₁, v₁, c₂, v₁, v₂, c₃, v₁, v₂, c₄, v₁, v₂) = -\beta Q v₂ v₃ c₂ c₁ \rho + \beta² Q v₂ v₃ c₂ c₁ \rho - \beta² v₂ v₃ c₂ c₁ \rho + \beta² Q v₂ v₃ c₂ c₁ \rho + v₂ β² c₂ c₁ c₃ c₁ \rho + \beta² v₂ v₂ c₂ c₁ c₃ c₁ \rho + v₂ β² v₂ v₂ c₂ c₁ c₃ c₁ \rho - Q v₂ v₂ c₂ c₁ c₃ c₁ \rho \]
\[ -Pv_1Rv_{12} + Q^2v_2v_{11} - Q^2v_2v_{13} + Pn_1Rv_{12} - v_2PcRv_{12}\rho - v_2v_1Rcc\rho \\
-2\beta v_1Rv_{12} + v_2Q\beta Qcv_{12}\rho - 2\beta v_1Rccv_{12}\rho - v_2Q\beta v_{12}Rcv_{12}\rho \\
+ \beta^2v_1Pcv_{12}\rho - 2\beta Qv_1Rcv_{12}\rho - \beta Qv_1Rccv_{12}\rho + \beta^2v_1Rcv_{12}\rho \\
+ \beta^2Qv_1Rcv_{12}\rho - 2\beta Qv_1Rccv_{12}\rho + \beta^2v_1Rcv_{12}\rho + \beta^2v_1Rcv_{12}\rho \\
\Theta(c, \rho, \beta, v_{11}, v_{12}, c_1, v_{11}, v_{12}) = -Qv_1c_1v_{11}\rho - 2\beta v_1c_1v_{11}\rho + \beta Qv_1c_1v_{11}\rho - v_2Q\beta v_{12}Rccv_{12}\rho + \beta^2Qv_1ccv_{12}\rho \\
+ \beta^2Q\beta Qcv_{12}\rho + 2\beta Qv_1Rcv_{12}\rho + 2\beta Qv_1Rccv_{12}\rho + Pn_1Rv_{12} - Q^2v_2v_{11} + Q^2v_2v_{11} \\
- Pn_1Rv_{12} - v_2\beta v_2Rcv_{12}\rho + v_2\beta^2v_2Rcv_{12}\rho + v_2\beta^2Rccv_{12}\rho - \beta Qv_1ccv_{12}\rho \\
+ 2\beta^2Qv_1ccv_{12}\rho - 2\beta^2Qv_1ccv_{12}\rho + \beta Qv_1ccv_{12}\rho - \beta Qv_2ccv_{12}\rho \\
- v_1Rcv_{12}\rho + v_2\beta^2v_1Rccv_{12}\rho - v_2v_1Rccv_{12}\rho + \beta Qv_2v_{13}c_1\rho \\
- Qv_1v_1ccv_{12}\rho + 2\beta v_2Rcv_{12}\rho + v_1v_1ccv_{12}\rho - v_2^2v_1v_{12}ccv_{13}\rho + 2\beta^2v_1v_1ccv_{12}\rho \\
- 2\beta^2Qv_1v_{12}ccv_{13}\rho - v_2^2Qv_1ccv_{12}\rho + 2\beta Qv_1v_{12}ccv_{13}\rho \\
\Xi(c, \rho, \beta, v_{11}, v_{12}, c_1, v_{11}, v_{12}) = -2c_1c(-v_1\beta + \beta v_{12} + v_{11})\rho(\beta Rv_{12} - v_{11}Q + \beta Pv_{13} - Rv_{12} + v_{13}Q + v_{13}Q) \\
\Gamma(c, \rho, \beta, v_{11}, v_{12}, c_1, v_{11}, v_{12}) = (2\beta(1 - Q^2)(-v_2v_1v_{12} + v_1v_1v_{12})\rho(\beta Rv_{12} - v_{11}Q + \beta Pv_{13} - Rv_{12} + v_{13}Q + \beta v_{13}Q) \\
+ \beta Pv_{13} - Rv_{12} + v_{13}Q + \beta v_{13}Q) \times c_1(-v_{11}\beta + \beta v_{12} + v_{11}).
\]

Appendix B. Numerical computation of the Laplace transform and its inverse

With \(s = \sigma_0 + i\omega\) the forward transform is computed as

\[
\hat{f}(s) = F(f(t)) = \int_0^\infty e^{-(\sigma_0+i\omega)t}f(t)dt \equiv \int_0^T e^{-(\sigma_0+i\omega)t}f(t)dt
\]

for large \(T\). The numerical value \(\sigma_0 = 0.1\) was used for all transforms in this article. Let the fast Fourier transform of a vector \(v = (v_n)\) be denoted by \(\text{fft}(v)\). Upon discretization

\[
\hat{f}_n = \frac{T}{N} \sum_{k=0}^{N-1} (e^{-2\pi iNkn/k}) e^{-\sigma_0kT/N} f_k = \frac{T}{N} \text{fft}(e^{-\sigma_0kT/N} f_k)_{k=0}^{N-1}, \quad n = 0, \ldots, N - 1
\]

is the approximation to \(\hat{f}(\sigma_0 + 2\pi in/T)\). The inverse transform is

\[
f(t) = \frac{1}{2\pi} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{it\hat{f}(s)}ds \equiv \frac{1}{2\pi} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{it\hat{f}(s)}ds
\]

for large \(\Omega\). Discretization and use of the inverse fast Fourier transform \(\text{ifft}(v)\) gives

\[
f_k = \frac{e^{2\pi\sigma_0k/\Omega}}{\pi N} \left(-\frac{1}{2} f_0 + \sum_{n=0}^{N-1} (e^{2\pi iNk/n}) f_n \right)
\]

\[
= \frac{e^{2\pi\sigma_0k/\Omega}}{\pi} \left(-\frac{1}{2N} f_0 + \text{ifft}(\hat{f}_n)_{n=0}^{N-1} \right), \quad k = 0, \ldots, N - 1
\]

for the approximate inverse at \(t_k = 2\pi k/\Omega\).

Appendix C. Parameters for specimens used in simulations

The parameters in Table 5 are those of the specimen M1 of [4] (Tables 6–9). The parameters in Table 10 are those of the specimen M3 of [5]. The following parameters were the same for all simulations: \(\rho_0 = \rho_1 = \rho_2 = 1000\ \text{kg}\ \text{m}^{-3}, K_f = K_0 = K_1 = 2.28E + 09\ \text{Pa}, \eta = 0.001\ \text{kg\ (ms)}^{-1}.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta)</td>
<td>0.83</td>
</tr>
<tr>
<td>(\sigma_0)</td>
<td>1.05</td>
</tr>
<tr>
<td>(\Lambda)</td>
<td>5.00E–06</td>
</tr>
<tr>
<td>(\rho_0)</td>
<td>1960</td>
</tr>
<tr>
<td>(K_f)</td>
<td>2.00E+10</td>
</tr>
<tr>
<td>(K_s)</td>
<td>2.6E+09</td>
</tr>
<tr>
<td>(K_s)</td>
<td>3.3E+09</td>
</tr>
</tbody>
</table>
Table 6
The parameters for specimen M2 of Fellah et al. [4].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.77</td>
<td></td>
</tr>
<tr>
<td>$\alpha_\infty$</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>2.70E−06</td>
<td>m</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>1970</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>$K_s$</td>
<td>2.00E+10</td>
<td>Pa</td>
</tr>
<tr>
<td>$G$</td>
<td>1.70E+09</td>
<td>Pa</td>
</tr>
<tr>
<td>$K_b$</td>
<td>4.00E+09</td>
<td>Pa</td>
</tr>
</tbody>
</table>

Table 7
The parameters for specimen M3 of Fellah et al. [4].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.88</td>
<td></td>
</tr>
<tr>
<td>$\alpha_\infty$</td>
<td>1.02</td>
<td></td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>5.00E−06</td>
<td>m</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>1980</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>$K_s$</td>
<td>2.60E+10</td>
<td>Pa</td>
</tr>
<tr>
<td>$G$</td>
<td>3.50E+08</td>
<td>Pa</td>
</tr>
<tr>
<td>$K_b$</td>
<td>1.30E+09</td>
<td>Pa</td>
</tr>
</tbody>
</table>

Table 8
The parameters for specimen M1 of Sebaa et al. [5].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.64</td>
<td></td>
</tr>
<tr>
<td>$\alpha_\infty$</td>
<td>1.018</td>
<td></td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>9.10E−06</td>
<td>m</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>1990</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>$K_s$</td>
<td>1.08E+10</td>
<td>Pa</td>
</tr>
<tr>
<td>$G$</td>
<td>1.75E+09</td>
<td>Pa</td>
</tr>
<tr>
<td>$K_b$</td>
<td>3.40E+09</td>
<td>Pa</td>
</tr>
</tbody>
</table>

Table 9
The parameters for specimen M2 of Sebaa et al. [5].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.79</td>
<td></td>
</tr>
<tr>
<td>$\alpha_\infty$</td>
<td>1.052</td>
<td></td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>1.01E−05</td>
<td>m</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>1990</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>$K_s$</td>
<td>1.08E+10</td>
<td>Pa</td>
</tr>
<tr>
<td>$G$</td>
<td>9.88E+08</td>
<td>Pa</td>
</tr>
<tr>
<td>$K_b$</td>
<td>1.65E+09</td>
<td>Pa</td>
</tr>
</tbody>
</table>

Table 10
The parameters for specimen M3 of Sebaa et al. [5].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.64</td>
<td></td>
</tr>
<tr>
<td>$\alpha_\infty$</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>1.50E−05</td>
<td>m</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>1990</td>
<td>kg m$^{-3}$</td>
</tr>
<tr>
<td>$K_s$</td>
<td>1.08E+10</td>
<td>Pa</td>
</tr>
<tr>
<td>$G$</td>
<td>1.27E+09</td>
<td>Pa</td>
</tr>
<tr>
<td>$K_b$</td>
<td>1.85E+09</td>
<td>Pa</td>
</tr>
</tbody>
</table>

References