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9 THE SEAMOUNT ON A SLOPING SEABED PROBLEM

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ABSTRACT

This paper deals with an inverse acoustics problem in the ocean. The problem we investigate is the location of a non-homogeneity caused by a sea-mount or some object lying on a sloping seabed. This problem is solved by constructing an acoustic Green's function for the wedge. This is done by using the method of images. The inversion procedure is motivated by our earlier work on the seamount problem for a shallow ocean of uniform depth [17].

1. FORMULATION OF THE DIRECT PROBLEM

Buchanan, Gilbert, Wirgin, and Xu have investigated, in detail, inverse problems for uniform, finite depth oceans with a completely reflecting basement. These results were reported on in a sequence of papers [2, 3, 12, 13, 16, 9, 10]. The methodology used in these papers was first to obtain an operator which produced the far field from an incident ray scattered off the target. Then the inverse problem was formulated as an extremal problem. With a suitable fundamental singular solution
for the wedge domain, this method is applicable to an ocean with an inclined seabed.

Across-section in the $x-y$ plane of a buried object on a sloping seabed is visualized in Figure 1. The $z$ axis is assumed to lie perpendicular to the page. It is assumed that the basement is completely reflecting. Then the acoustic pressure, generated by a point source at the given location $\vec{x} = (x_0, y_0, z_0)$, satisfies

$$\Delta p + k^2 p = -\delta(\vec{x} - \vec{x}_0), \quad \vec{x} \in \mathbb{R}^3 \setminus \Omega,$$

$$p = 0 \quad \text{at} \quad y = 0$$

$$\frac{\partial p}{\partial z} = 0 \quad \text{at} \quad x = y \tan(\theta_0), \quad (x, y) \not\in \mathcal{M}$$

$$\frac{\partial p}{\partial n} = 0 \quad \text{on} \quad \mathcal{M}.$$

![Figure 1. Cross-section of an object partially buried on an inclined seabed.](image)

And the Sommerfeld out-going radiation condition. The wedge region we consider is

$$\mathbb{R}^3_b = \left\{(x, y, z) : 0 \leq x < \infty, \quad 0 \leq y \leq x \tan(\theta_0), \quad |z| < \infty\right\}$$

or in cylindrical coordinates as

$$\{(r, \theta, z) : 0 \leq r < \infty, \quad 0 \leq \theta \leq \theta_0, \quad |z| < \infty\};$$

$\Omega$ is the sea-mount, and $\mathcal{M}$ is the surface of the sea-mount,

$$\mathcal{M} : = \left\{(x, y, z) : y = f(x, z), \quad \text{where} \quad (x, z) \in D_b\right\}.$$

Here $D_b$ is the projection of the sea-mount $D$ onto the plane $y = 0$. The sea-mount is denoted by $D : = \left\{(x, y, z) : x \tan(\theta_0) < y < f(x, z), \quad \text{where} \quad (x, z) \in D_b\right\}$.

For an ocean with a sloping seabed without a sea-mount, the solution to (1.1), (1.2), (1.3), and (1.4) is the Green's function for the Helmholtz equation in $\mathbb{R}^3_b$.

Buchingham [4, 5] has constructed this Green's function using integral transforms;
however, for our purposes, we need to exhibit clearly the singular behavior at the source point. To this end we construct the Green's function by the method of images. Suppressing the z variable, which is perpendicular to the wedge, we begin with a source at the point \((x_0, y_0)\). There are two sequences of images that we obtain. The first sequence begins by a reflection through the line \(x = y \tan(\theta_0)\) and then proceeds with a reflection through \(y = 0\) another through \(x = y \tan(\theta_0)\), etc. We designate these source points as \((x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\). The second sequence begins with a reflection through \(y = 0\), the next through \(x = y \tan(\theta_0)\), the third reflection through \(y = 0\), etc. We designate these source points as \((x_0, y_0), (\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2), \ldots\). We construct these image points by successive reflections and rotations as follows. Let \(T_{\theta_0} x \rightarrow x'\) be the transformation from the \((x, y)\) coordinate frame to the \((x', y')\) coordinate frame. It is represented by the rotation through the angle \(\theta_0\)

\[
T_{\theta_0} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta_0) & -\sin(\theta_0) \\ \sin(\theta_0) & \cos(\theta_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Figure 2. The sequence of image points to compute the Green's function.
The inverse of $T_{\theta_0}$ is the transformation from the $(x', y')$ coordinate frame to the $(x, y)$ coordinate frame. It is represented by $T_{-\theta_0}$, $x' \rightarrow x$, that is the rotation through the angle $-\theta_0$

$$T'(x', y') = \begin{pmatrix} \cos(\theta_0) & \sin(\theta_0) \\ -\sin(\theta_0) & \cos(\theta_0) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Let $E$ be the reflection matrix

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The image point $(x_i, y_i)$ is then computed by the scheme

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_0) & \sin(\theta_0) \\ -\sin(\theta_0) & \cos(\theta_0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta_0) & -\sin(\theta_0) \\ \sin(\theta_0) & \cos(\theta_0) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = E T_{2\theta_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$ 

In a similar manner we compute $(x_2, y_2)$ to be

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = E E T_{2\theta_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = T_{4\theta_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$ 

The image point $(x_3, y_3)$ is also found by this procedure to be

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = E T_{4\theta_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$ 

It may be shown inductively that the image points $(x_n, y_n)$, $n = 1, 2, 3, \ldots$ are given for even values $n = 2k$ by

$$\begin{pmatrix} x_{2k} \\ y_{2k} \end{pmatrix} = T_{2k\theta_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix};$$

whereas, for odd values $n = 2k + 1$ they are

$$\begin{pmatrix} x_{2k+1} \\ y_{2k+1} \end{pmatrix} = E T_{(2k+1)\theta_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$ 

We next consider the image points of the sequence $(\bar{x}_n, \bar{y}_n)$. It may be shown that these points are given for $n = 2k$ by

$$\begin{pmatrix} \bar{x}_{2k} \\ \bar{y}_{2k} \end{pmatrix} = \begin{pmatrix} \cos(2k\theta_0) & \sin(2k\theta_0) \\ -\sin(2k\theta_0) & \cos(2k\theta_0) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

and for $n = 2k + 1$

$$\begin{pmatrix} \bar{x}_{2k+1} \\ \bar{y}_{2k+1} \end{pmatrix} = E \begin{pmatrix} \bar{x}_{2k} \\ \bar{y}_{2k} \end{pmatrix} = \begin{pmatrix} \cos(2k\theta_0) & \sin(2k\theta_0) \\ \sin(2k\theta_0) & -\cos(2k\theta_0) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$ 

Consequently, the Green’s function has the representation
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\[ G(\vec{x}, \vec{x}_0) + E(\vec{x}, \vec{x}_0) + \sum_{n=0}^{\infty} (-1)^n \left[ E(\vec{x}, \vec{x}_n) - E(\vec{x}, \vec{x}_{n+1}) \right], \tag{1.5} \]

where

\[ \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{and} \quad \vec{x} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \]

and

\[ E(\vec{x}, \vec{y}) = \frac{\exp\left\{ i k (|\vec{x} - \vec{y}|) \right\}}{4\pi|\vec{x} - \vec{y}|}. \tag{1.6} \]

The solution of problem (1.4) may be represented using Green’s formula as

\[ p(\vec{x}, \vec{x}_0) = G(\vec{x}, \vec{x}_0) + \int_{\mathcal{M}} \left[ G(\vec{x}, \vec{y}) \frac{\partial \rho_n(\vec{y})}{\partial\nu} - \rho_n(\vec{y}) \frac{\partial G(\vec{x}, \vec{y})}{\partial\nu} \right] d\mathcal{S}_y, \tag{1.7} \]

for \( \vec{x} \in \mathbb{R}^3 \setminus \bar{\Omega} \); here \( \rho_n(\vec{y}) \) is the unique solution of the integral equation

\[ \rho_n(\vec{y}) + 2 \int_{\mathcal{M}} \rho_n(\vec{x}) \frac{\partial G(\vec{x}, \vec{y})}{\partial\nu} d\mathcal{S}_x = -2 \int_{\mathcal{M}} G(\vec{x}, \vec{y}) \frac{\partial}{\partial\nu} G(\vec{x}, \vec{x}_0) d\mathcal{S}_x, \quad \vec{y} \in \mathcal{M} \tag{1.8} \]

and

\[ \frac{\partial}{\partial\nu} \rho_n(\vec{y}) = -\frac{\partial}{\partial\nu} G(\vec{x}, \vec{x}_0), \quad \vec{y} \in \mathcal{M}. \tag{1.9} \]

The inverse problem consists of determining the sea-mount \( \mathcal{M} \) when \( p(\vec{x}, \vec{x}_0) \) is given for all \( \vec{x} \in \Gamma_1 \cap \mathbb{R}^3 \), \( \Gamma_1 := \{ (x,y,z) : z = d_1 = \text{constant} \} \), and \( \vec{x}_0 \in \Gamma_2 \cap \mathbb{R}^3 \), \( \Gamma_2 := \{ (x,y,z) : z = d_2 = \text{constant} \} \).

Here we assume that \( \Gamma_1 \) and \( \Gamma_2 \) are strictly above the sea-mount, i.e.,

\[ \max_{x} \left\{ z = f(x,y) \right\} < \min \{ d_1, d_2 \}. \]

1.1 Uniqueness of the sea-mount problem

We assume that both \( \Gamma_1 \cap \mathbb{R}_0^3 \) (the receiving plane) and \( \Gamma_2 \cap \mathbb{R}_0^3 \) (the source location plane) are above the sea-mount. That is, \( \mathcal{M} \) is disjoint with \( \Gamma_j \cap \mathbb{R}_0^3 \), \( j = 1, 2 \). The proofs of the following theorems are similar in structure to the approach we used in [17]; hence, we refer the reader to that work for further details.

**Theorem 1:** Assume that \( D_1 \) and \( D_2 \) are two sea-mounts with rigid boundaries \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) respectively. Furthermore, suppose that the corresponding solutions of problem (1.4) coincide on \( \Gamma_1 \cap \mathbb{R}_0^3 \) for all \( \vec{x}_0 \in \Omega \), where \( \Omega \) is the unbounded component of \( \mathbb{R}_0^3 \cup (\bar{D}_1 \cup \bar{D}_2) \), then \( D_1 = D_2 \).
In general, one sends in incident waves from all directions. In order to simplify this requirement, we need the following lemmas:

**Lemma 1:** Let \( D \subset R^n_+ \) be a bounded domain with \( C^2 \) boundary and assume that \( R^n_+ \setminus D \) is connected. \( D \) is located strictly below \( \Gamma_1 \cap R^n_+ \), i.e., \( \min \{|x,y,z| \in D\} > d_1 \). Let \( G(\cdot, \bar{x}_0) \) be the Green’s function for the wedge with source at \( \bar{x}_0 \),

\[
H := \left\{ \frac{\partial G}{\partial \nu} (\cdot, \bar{x}_0) - i G(\cdot, \bar{x}_0) : \bar{x}_0 \in \Gamma_1 \right\}.
\] (1.10)

Then \( H \) is complete in \( L^2(\partial D) \).

**Proof:** Assume \( \varphi \in L^2(\partial D) \) satisfies

\[
\int_{\partial D} \varphi(\bar{y}) \left[ \frac{\partial}{\partial \nu} G(\bar{y}, \bar{x}_0) - i G(\bar{y}, \bar{x}_0) \right] d\sigma(\bar{y}) = 0,
\] (1.11)

for all \( \bar{x}_0 \in \Gamma_1 \). Then the combined single- and double-layer potential

\[
u(\bar{x}) := \int_{\partial D} \varphi(\bar{y}) \left[ \frac{\partial}{\partial \nu} G(\bar{y}, \bar{x}) - i G(\bar{y}, \bar{x}) \right] d\sigma(\bar{y}), \quad x \in R^n_+ \setminus \partial D,
\] (1.12)

satisfies the Helmholtz equation in \( R^n_+ \setminus D \), the out-going radiation condition as \( r \to \infty \), vanishing Neumann data on \( x = y \tan \theta_0 \), \( y > d_1 \), and

\[\nu(\bar{x}) \bigg|_{r = \kappa} = \nu(\bar{x}) \bigg|_{r = 0} = 0.
\] (1.13)

It implies that \( \nu = 0 \) in \( R^n_+ \setminus \bar{D} \), as shown in [17]. Because the singularity of the Green’s function is due to the first term in its series expansion we may group the Green’s function as

\[G(\bar{x}, \bar{y}) = \frac{e^{ik|\bar{x} - \bar{y}|}}{4\pi|\bar{x} - \bar{y}|} + \Phi_1(\bar{x}, \bar{y}),
\] (1.14)

where \( \Phi_1(\bar{x}, \bar{y}) \) is continuous at \( \bar{x} \to \bar{y} \), we obtain the boundary integral equation

\[\varphi + K\varphi - iS\varphi = 0 \quad \text{on} \quad \partial D.
\] (1.15)

Here

\[
K \varphi(\bar{x}) := 2 \int_{\partial D} \frac{\partial G}{\partial \nu} (\bar{y}, \bar{x}) \varphi(\bar{y}) d\sigma(\bar{y}),
\] (1.16)

\[
S \varphi(\bar{x}) := 2 \int_{\partial D} G(\bar{y}, \bar{x}) \varphi(\bar{y}) d\sigma(\bar{y}).
\] (1.17)

The operator \( 1 + K - iS \) is invertible in the wedge and its inverse is a bounded linear operator in \( L^2(\partial D) \). Hence, we have from (1.9) \( \varphi = 0 \) on \( \partial D \) and the completeness of \( H \) is proved.
Lemma 2: Let $D$ be a bounded domain with $C^2$ boundary $\partial D$ such that $R^3 \setminus \overline{D}$ is connected. $D$ is located strictly below $\Gamma$. Let $u \in C^2(D) \cap C^1(\partial D)$ be a solution of the Helmholtz equation. Then there exists a sequence $\{u_n\}$ in the span of

$$V := \text{span} \{ G(\cdot, \tilde{x}_n) : \tilde{x}_n \in \Gamma \}$$

such that

$$u_n \to u, \quad \nabla u_n \to \nabla u, \quad \text{as} \quad n \to \infty, \quad (1.14)$$

uniformly on compact subsets of $D$.

Theorem 2: Assume that $D_1$ and $D_2$ are two seamounts with rigid boundaries $M_1$ and $M_2$, such that the corresponding solutions of (1.4) coincide on $\Gamma$, for all $x_0 \in \Gamma$, then $D_1 = D_2$.

In view of the $u_n$ being linear combinations of point-source waves from sources on $\Gamma$, are the assumption of the Theorem it follows that the corresponding solutions $u_{n,1}$ and $u_{n,2}$ for the seamounts $D_1$ and $D_2$ coincide in $\Gamma$. Using the same argument as is used in the proof of Theorem (1) (see [17]) it follows that

$$u_n^j := u_{n,1}^j = u_{n,2}^j \quad \text{in} \quad \Omega. \quad (1.15)$$

Moreover,

$$\frac{\partial v_n^j}{\partial \nu} + \frac{\partial u_n^j}{\partial \nu} = 0 \quad \text{on} \quad \partial D_j \cap \partial \Omega, \quad j = 1, 2. \quad (1.16)$$

As a consequence of the continuous dependence of the solution to the exterior Neumann problem on the boundary condition, along with the boundary condition (1.24) and the convergence (1.22), it follows that

$$u_n^j \to p^j, \quad n \to \infty, \quad (1.17)$$

uniformly on compact subsets of $\Omega$ for $j = 1, 2$. Therefore, it must hold that $p_1^j = p_2^j$ in $\Omega$. By Theorem 1, we conclude that $D_1 = D_2$.

1.2 A linearized algorithm for construction of the seamount

Let us consider the following linearized algorithm to find the shape of the seamount. Let $f_0(x,y)$ be an initial guess for the shape function $f(x,y)$. Then we propose the following recursion scheme to determine the shape function:

$$\delta p_n = p - p_n, \quad \text{and} \quad \delta f_n = f - f_n, \quad n = 0, 1, 2, \ldots, \quad (1.18)$$

and where the corresponding sequence of the partially submerged object and its surface are given by

$$\mathcal{D}_n := \{ (x,y,z) : x \tan(\theta_n) > y > f_n(x,z), (x,y) \in D_n \}$$

$$\mathcal{M}_n := \{ (x,y,z) : y = f_n(x,z), (x,z) \in D_n \}.$$
Substituting (1.26) into (1.1-1.4) and neglecting terms of $O(\delta^3)$ and higher we have

$$\Delta p_n + k^2 p_n = -\delta(\bar{x} - \bar{x}_0), \quad \text{where} \quad \bar{x} \in \mathcal{R}_q \setminus \overline{D_q},$$

$$p_n = 0 \quad \text{at} \quad y = 0$$

$$\frac{\partial p_n}{\partial v} = 0 \quad \text{at} \quad y = x \tan(\theta_0), \quad (x, y) \in \mathcal{M}_n$$

$$\frac{\partial p_n}{\partial v} = 0 \quad \text{on} \quad \mathcal{M}_n, \ n = 0, 1, \ldots,$$

and

$$\Delta \delta p_n + k^2 \delta p_n = 0, \quad \text{where} \quad \bar{x} \in \mathcal{R}_q \setminus \overline{D_q}$$

$$\delta p_n = 0 \quad \text{at} \quad y = 0$$

$$\frac{\partial \delta p_n}{\partial v} = 0 \quad \text{at} \quad x = y \tan(\theta_0), \quad (x, y) \in \mathcal{M}_n$$

$$\frac{\partial \delta p_n}{\partial v} = -\left(\frac{\partial^2}{\partial v^2} p_n\right) \delta f_n, \quad \text{on} \quad \mathcal{M}_n.$$

We can now use single-layer potentials to obtain a relation between $\delta p_n$ and $\delta f_n$ in (1.32) – (1.34). Let us represent $\delta p_n$ as

$$\delta p_n(\bar{x}) = \int_G(\bar{x}, \bar{y}) \phi(\bar{y}) \, ds, \quad \bar{x} \in \mathcal{R}_q \setminus \overline{D_q},$$

Then $\phi(\bar{y})$ satisfies

$$\phi(\bar{x}) - 2 \int_{\mathcal{M}_n} \frac{\partial G(\bar{x}, \bar{y})}{\partial v} \phi(\bar{y}) \, ds = -2 \frac{\partial^2}{\partial v^2} p_n \delta f_n \text{ on } \mathcal{M}_n,$$

and

$$\int_G(\bar{x}, \bar{y}) \phi(\bar{y}) \, ds = \delta p_n(\bar{x}) = p(\bar{x}) - p_n(\bar{x}), \quad \text{for} \quad \bar{x} \in \Gamma_{1}.$$

This suggests an iterative algorithm for solving the inverse problem:

1. Make an initial guess for the shape function $f_0(\bar{x}, z)$.
2. At the $n$th stage, solve for $p_n(\bar{x})$ using (1.27) – (1.30).
3. Using $p(\bar{x}) - p_n(\bar{x})$ for $\delta p_n(\bar{x})$, solve $\phi(\bar{y}) = \phi_n(\bar{y})$ for $\bar{y} \in \mathcal{M}_n$ using (1.37).
4. For a chosen accuracy, given by $\varepsilon > 0$, define

$$\delta f_n := \min \left\{ \varepsilon, \left[ \phi(\bar{x}) - 2 \int_{\mathcal{M}_n} \frac{\partial G(\bar{x}, \bar{y})}{\partial v} \phi(\bar{y}) \, ds, \left[-2 \frac{\partial^2}{\partial v^2} p_n\right]^{-1}\right] \right\}.$$

5. Now upgrade $f_n + \delta f_n \rightarrow f_{n+1}$.
6. We repeat the above steps for $n = 1, 2, \ldots$ solving for $p_n, \delta p_n, \phi_n, \delta f_n$ respectively until $|\delta f_n| < \varepsilon$ for some chosen $\varepsilon$.\n
Step 3 in the above algorithm solves an ill-posed integral equation, inherited from the original ill-posedness of the inverse problem. A proper regularization method must be adapted in order to solve (1.37). With this in mind, we first discuss some properties of the integral operators $T$ and $T_n$, defined by

$$T\phi(\bar{x}) = \int_{\mathfrak{M}} G(\bar{x}, \bar{y}) \phi(\bar{y}) \, ds_\bar{y}, \quad \forall \bar{y} \in \mathfrak{M},$$

$$T_n \phi(\bar{x}) = \int_{\mathfrak{M}} G(\bar{x}, \bar{y}) \phi(\bar{y}) \, ds_\bar{y}, \quad \bar{x} \in \Gamma, \quad (n = 0, 1, 2, \ldots).$$

We will need the following spaces that are weighted in $x = (x_1, x_2) \in \mathbb{R}^2$,

$$L^2(\mathbb{R}^2) = \left\{ u : \left[ 1 + |x|^2 \right]^{\nu/2} u \in L^2(\mathbb{R}^2) \right\},$$

$$H^{1,1}(\mathbb{R}^2) = \left\{ u : D^\alpha u \in L^2(\mathbb{R}^2), |\alpha| < 1 \right\},$$

where we use the multi-index notation $\alpha = (\alpha_1, \alpha_2), |\alpha| = |\alpha_1| + |\alpha_2|$ and $D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$; $L^2$ denote the space of square-integrable functions. We use $L^2(\mathcal{M}), H^1(\mathcal{M}), L^2(\mathcal{M}_n)$ and $H^1(\mathcal{M}_n)$ to denote the usual Hilbert spaces and Sobolev spaces.

In view of the normal mode integral [4, 5] representation of $G(\bar{x}, \bar{y})$

$$G(\bar{x}, \bar{y}) = \sum_{\ell=0} \int \frac{L_{\ell}(r, r', z) \sin(\nu \theta) \sin(\nu \theta')}{n},$$

where the modal integral is defined by

$$L_{\ell}(r, r', z) = \int_0^\frac{\pi}{2} \frac{\exp{-\eta |\ell|}}{\eta} J_\ell(\nu r) J_\ell(\nu r') \, dp,$$

where $\eta = \sqrt{p^2 - k^2}, \nu = \frac{(m + 1/2) \pi}{\theta_0}$ and

$$J_\ell(\zeta) \text{ is a Bessel function of the first kind and order } \nu.$$ From this representation, we know that $G(\bar{x}, \bar{y})$ is real analytic in $\bar{x}$ for any $\bar{y} \in \mathfrak{M}$; and for some constant $C$,

$$|G(\bar{x}, \bar{y})| < C |x|^{-\nu/2},$$

$$|D^\alpha G(\bar{x}, \bar{y})| < C |x|^{1-\nu/2}, \quad |\alpha| \leq 2,$$

uniformly for $\bar{y} \in \mathfrak{M}$ as $|x| \to \infty$. From this it follows:

**Theorem 3:**

1. The operator $T$ is compact from $L^2(\mathfrak{M})$ into $H^{1,1}(\Gamma)$ for $s > 1/2$.
2. The operator $T_n$ is compact from $L^2(\mathfrak{M}_n)$ into $H^{1,1}(\Gamma_n)$ for $s > 1/2$.
Theorem 4: The operator $T$ is injective and has dense range provided that the mixed boundary valued problem
\begin{align}
\Delta u + k^2 u &= 0, \quad \bar{x} \in \overline{D}, \\
u &= 0 \quad \text{on } \mathcal{M}, \\
\frac{\partial u}{\partial v} &= 0 \quad \text{at } \left\{ x = y \tan(\theta_a) \right\} = : \gamma_{\theta_a}, \quad 0 \leq x \leq a,
\end{align}
has no nontrivial solution.

Proof: We first prove that from $T \phi = 0$ it follows that $\phi = 0$. Consider
\begin{equation}
u(\bar{x}) = \int_{\mathcal{M}} g(\bar{x}, \bar{y}) \phi(\bar{y}) \, ds_y, \quad \bar{x} \in \mathbb{R}^3.
\end{equation}
$u(\bar{x})$ satisfies (1.23) in $\mathbb{R}^3 \setminus \overline{D}$, $u(\bar{x}) = 0$ for $\bar{x} \in \Gamma_1 \cup \Gamma_a$ and $u(\bar{x})$ satisfies the outgoing radiation condition. Then $u(\bar{x}) = 0$ for $\bar{x} \in \left\{ \{r, \theta, z\} : \Gamma_0 : = \{y = 0\} \right\}$; hence, $u(\bar{x}) = 0$ for $\bar{x} \in \mathbb{R}^3 \setminus \overline{D}$.

Define
\begin{equation}
K \phi(\bar{x}) = 2 \int_{\mathcal{M}} \frac{\partial}{\partial v_y} G(\bar{x}, \bar{y}) \phi(\bar{y}) \, ds_y, \quad \bar{x} \in \mathcal{M}.
\end{equation}
The jump relation of the normal derivative of $u(\bar{x})$ on $\mathcal{M}$ implies
\begin{equation}
\phi - K \phi = 0, \quad \text{on } \mathcal{M}.
\end{equation}
A typical way similar to that in ([7], p128) show that $\phi = 0$ on $\mathcal{M}$, provided the problem (1.34)-(1.37) has no nontrivial solution. We conclude that $T$ is injective.

Now we show that if $(\psi, T \phi)_{L^2(\mathcal{M})} = 0$ for all $\phi \in L^2(\mathcal{M})$, then $\psi = 0$. That is, we need to show that from $T' \psi = 0$ on $\mathcal{M}$ it follows that $\psi = 0$ on $\Gamma$; here $T' : L^2_{-\gamma}(\Gamma_1) \to L^2(\mathcal{M})$ is the adjoint operator of $T$:
\begin{equation}
T' \psi(\bar{y}) = \int_{\Gamma} G(\bar{x}, \bar{y}) \psi(\bar{x}) \left( \left| x_1 \right|^2 + 1 \right)^{-\gamma/2} \, ds_x, \quad \bar{y} \in \mathcal{M}.
\end{equation}
Now consider
\begin{equation}
u(\bar{y}) = \int_{\Gamma} G(\bar{x}, \bar{y}) \psi(\bar{x}) \left( \left| x_1 \right|^2 + 1 \right)^{-\gamma/2} \, ds_x, \quad \bar{y} \in \mathbb{R}^3.
\end{equation}
$\nu(\bar{y})$ is a solution of the problem (1.34)-(1.37); hence, $\nu(\bar{y}) = 0$ in $\Omega$. But $\nu(\bar{y})$ satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \Gamma_1$. So $\nu(\bar{y}) = 0$ on $\left\{ \{r, \theta, z\} : 0 \leq z \leq a \right\} \cap \mathbb{R}^3$. Define
\begin{equation}
S \psi(\bar{y}) = 2 \int_{\Gamma} \frac{\partial}{\partial v_x} G(\bar{x}, \bar{y}) \psi(\bar{x}) \left( \left| x_1 \right|^2 + 1 \right)^{-\gamma/2} \, ds_x, \quad \bar{y} \in \Gamma_1.
\end{equation}
The jump relation of the normal derivative of $\nu(\bar{y})$ on $\Gamma_1$ implies
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\[ \psi + S \psi = 0, \quad \text{on } \Gamma, \quad \text{(1.44)} \]

Now we can conclude similar to the discussion for \( u(\bar{x}) \) that \( \psi = 0 \) on \( \Gamma \).

Based on Theorems 3 and 4, we may apply the Tikhonov regularization to step 3, that is, we solve

\[ \alpha \phi_n + T_n^* T_n \phi_n = T_n^* \left( \rho - \rho_n \right) \quad \text{(1.45)} \]

with some regularization parameter \( \alpha > 0 \) instead of (1.29). The regularity of discrepancy principle for the Tikhonov regularization (see, for example, [7], Th. 4.16, p99) follows.

**Theorem 5:** If \( \delta \rho_n \in T \left( L^2 \left( \mathbb{R} \right) \right) \), then

\[ \phi_n = (\alpha I + T_n^* T_n)^{-1} T_n^* \left( \rho - \rho_n \right) \quad \text{(1.46)} \]

approaches \( T_n^{-1} \delta \rho_n \) as \( \alpha \to 0 \).

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**REFERENCES**


