On the integral representation formula for a two-component elastic composite

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SUMMARY

The aim of this paper is to derive, in the Hilbert space setting, an integral representation formula for the effective elasticity tensor for a two-component composite of elastic materials, not necessarily well-ordered. This integral representation formula implies a relation which links the effective elastic moduli to the $N$-point correlation functions of the microstructure. Such relation not only facilitates a powerful scheme for systematic incorporation of microstructural information into bounds on the effective elastic moduli but also provides a theoretical foundation for inverse-homogenization. The analysis presented in this paper can be generalized to an $n$-component composite of elastic materials. The relations developed here can be applied to the inverse-homogenization for a special class of linear viscoelastic composites. The results will be presented in another paper. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: effective elasticity tensor; microstructure; integral representation formula

1. INTRODUCTION

The systematic study of bounds on the effective elasticity tensor of composites can be traced back to the seminal paper by Hashin and Shtrikman [1] using variational methods. An approach based on analytic continuation, which does not rely on the availability of variational principles, was proposed by Kantor and Bergman [2]. In that paper, the integral representation formula of the effective elasticity tensor was used for systematic evaluation of bounds. Using this formula, the effective elasticity tensor is regarded as a linear functional on the set of positive finite Borel
measures on the interval \([0, 1]\) with prescribed moments which are related to the microstructure of the composite. The estimation of bounds is then formulated as an optimization problem over the set of Borel measures with constrained moments, see Reference [3]. Bounds of all orders, including classical bounds such as the Voigt bound, the Reuss bound and the Hashin–Shtrikman bound, can be derived using this approach. In the context of electrostatics, the integral representation formula for the effective dielectric tensor of a mixture of two isotropic materials was derived in Reference [3] in the Hilbert space setting assuming random stationary fields. The derivation of the integral representation formula in the Hilbert space setting for elastic composites with holes or hard inclusions can be found in Reference [4]. For the elastostatics problem presented in this paper, we derive the integral representation formula in the Hilbert space setting, the fields are deterministic without periodicity or statistical assumptions.

The paper is organized as follows: In Section 2, the elastostatics of the two-component composites is formulated as a boundary value problem (BVP) with affine boundary conditions. The existence and uniqueness of this BVP is analysed in Section 3. In Section 4, the mathematical definition of effective elasticity tensor is given and the integral representation formula in terms of a positive Borel measure is derived. The derivation of relations between the effective elasticity moduli and the moments of the Borel measure, together with the calculation of the moments from information on the microstructure are presented in Section 5. We would like to remark that the relations between the microstructure of the composite and the moments of the measure \(\lambda\) not only facilitate a powerful scheme for deriving bounds of effective elastic moduli of the composite [2], but also provide a theoretical foundation for inverse-homogenization. In Reference [5], the inverse-homogenization approach has been applied to isotropic dielectric composites. The relations developed here can be applied to the inverse-homogenization for a special class of linear viscoelastic composites. The results will be presented in another paper.

2. ELASTOSTATICS OF A TWO-COMPONENT COMPOSITE

We consider a composite of elastic material I and elastic material II, not necessarily well-ordered. Let \(C^{(1)}\) and \(C^{(2)}\) be the elasticity tensors for materials I and II, respectively, such that

\[
\begin{align*}
\alpha_1 \bar{\varepsilon} : \varepsilon &\leq \bar{\varepsilon} : C^{(1)} : \varepsilon \leq \beta_1 \bar{\varepsilon} : \varepsilon \\
\alpha_2 \bar{\varepsilon} : \varepsilon &\leq \bar{\varepsilon} : C^{(2)} : \varepsilon \leq \beta_2 \bar{\varepsilon} : \varepsilon
\end{align*}
\]

for some positive constants \(\alpha_1, \alpha_2, \beta_1, \beta_2\). Here and in the rest of the paper, \(\bar{\varepsilon}\) denotes the complex conjugate of \(\varepsilon\) and the symbol \(:\) is the operation of tensor contraction defined as

\[
\varepsilon : \eta \overset{\text{def}}{=} \sum_{i,j=1}^{3} \varepsilon_{ij} \eta_{ij}
\]

\[
\varepsilon : C : \eta \overset{\text{def}}{=} \sum_{i,j,k,l=1}^{3} \varepsilon_{ij} C_{ijkl} \eta_{kl}
\]

for any second-order tensors \(\varepsilon, \eta\) and fourth-order tensor \(C\).
A composite is said to be well-ordered if \( C^{(2)} \geq C^{(1)} \), i.e. \((\bar{\varepsilon} : C^{(2)} : \varepsilon) \geq (\bar{\varepsilon} : C^{(1)} : \varepsilon)\) for all symmetric second-order tensors \( \varepsilon \). Let \( M \) be the elastic tensor of a reference media such that there exist positive constants \( \beta_{M1}, \beta_{M2} \) and non-negative constants \( \gamma_{M1}, \gamma_{M2} \), \( \gamma_{M1} + \gamma_{M2} > 0 \) and

\[
\begin{align*}
\gamma_{M1} \bar{\varepsilon} : \varepsilon &\leq \bar{\varepsilon} : (M - C^{(1)}) : \varepsilon \leq \beta_{M1} \bar{\varepsilon} : \varepsilon \quad (3) \\
\gamma_{M2} \bar{\varepsilon} : \varepsilon &\leq \bar{\varepsilon} : (M - C^{(2)}) : \varepsilon \leq \beta_{M2} \bar{\varepsilon} : \varepsilon \quad (4)
\end{align*}
\]

for all symmetric second-order tensors \( \varepsilon \). For a well-ordered system, we may replace \( M \) with the larger tensor \( C^{(2)} \).

Following [2], we consider a class of composites whose elasticity tensor \( A \) has the following parametrization form:

\[
A(x;z) = M + (1-z)\chi_1(x)(C^{(1)} - M) + (1-z)\chi_2(x)(C^{(2)} - M)
\]

where \( \chi_1, \chi_2 \) are the characteristic functions of the region occupied by materials I and II, respectively. Note that the composite of interest corresponds to \( z = 0 \).

Let \( \Omega_1 \) and \( \Omega_2 \) be the bounded domain occupied by materials I and II, respectively. We assume both \( \Omega_1 \) and \( \Omega_2 \) have strictly positive measures. Define \( \Omega \equiv \Omega_1 \cup \Omega_2 \) and let \( W^{1,2}(\Omega) \) be the Sobolev space of \( L^2(\Omega) \) functions whose derivatives are also in \( L^2(\Omega) \). The closure of \( C^\infty_0(\Omega) \) in \( W^{1,2}(\Omega) \) is denoted by \( W^{1,2}_0(\Omega) \). We consider the following BVP:

\[
\begin{align*}
\varepsilon_{ij} &= \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), \quad i,j = 1 \ldots 3 \\
(\text{Div}(A(x;z) : \varepsilon))_i &= 0, \quad i = 1 \ldots 3 \\
u_i|_{\partial \Omega} &= \sum_{j=1}^{3} \epsilon_{ij} x_j, \quad i = 1 \ldots 3 \quad (6)
\end{align*}
\]

with the transmission conditions of continuity of the displacement \( u \) and the normal stress across the interface between \( \Omega_1 \) and \( \Omega_2 \). Here \( \epsilon^0 \) is a constant symmetric tensor and \( \text{Div} \) denotes the divergence operator with respect to \( x \in \mathbb{R}^3 \) defined as

\[
(\text{Div} A)_k = \sum_{i=1}^{3} \frac{\partial A_{ki}}{\partial x_i}, \quad k = 1 \ldots 3
\]

for any second-order tensor \( A \). The parameter \( z \) is any complex number in \( \mathbb{C} \). For most of \( z \), the BVP has a unique solution in \( (W^{1,2}(\Omega))^3 \). This will be made clear in the next section.

3. EXISTENCE AND UNIQUENESS OF SOLUTION TO BVP IN \( (W^{1,2}(\Omega))^3 \)

We note that in BVP, the integral of \( \varepsilon \) in BVP over \( \Omega \) equals \( \epsilon^0 \) for every \( z \) because of the affine boundary condition (6) and the continuity of \( u \) on the interface. Motivated by this, we write the strain tensor \( \varepsilon \) as

\[
\varepsilon = \epsilon^0 + \epsilon'
\]
and define the Hilbert space $\mathcal{E}$:

$$\mathcal{E}_0 \overset{\text{def}}{=} \left\{ \epsilon(x) : \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), u \in (C^1_0(\Omega))^3 \right\}$$

$$\mathcal{E} \overset{\text{def}}{=} \text{closure of } \mathcal{E}_0 \text{ in } (L^2(\Omega))^9$$

In this setting, the variational formulation of BVP is

$$\int_\Omega \tilde{\epsilon} : \Lambda (e^0 + e') \, dx = 0 \quad \forall e \in \mathcal{E}$$

(7)

where $\tilde{\epsilon}$ denotes the complex conjugate of $\epsilon$. For convenience, we define a new parameter $s$:

$$s \overset{\text{def}}{=} \frac{1}{1 - z}$$

**Theorem 3.1**

The BVP, in terms of the parameter $s$,

$$\begin{cases} 
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), & i,j = 1 \ldots 3 \\
\text{Div} \left[ M + \frac{1}{s} \chi_1(x)(C^{(1)} - M) + \frac{1}{s} \chi_2(x)(C^{(2)} - M) \right] : \epsilon_i = 0, & i = 1 \ldots 3 \\
u_i|_{\partial \Omega} = \sum_{j=1}^{3} \epsilon^0_{ij} x_j, & i = 1 \ldots 3
\end{cases}$$

has a unique solution $u \in (W^{1,2}(\Omega))^3$ for $s \in \mathbb{C} \setminus [0,1]$.

**Proof**

We note that the sesquilinear form

$$B(\epsilon, \epsilon') \overset{\text{def}}{=} \int_\Omega \tilde{\epsilon} : \Lambda (x;z) : \epsilon' \, dx$$

(8)

is bounded in the Hilbert space $\mathcal{E} \times \mathcal{E}$ for any fixed $z$. Also note in terms of $s$, $\Lambda$ has the following form:

$$\Lambda(x;s) = M + \frac{1}{s} \chi_1(x)(C^{(1)} - M) + \frac{1}{s} \chi_2(x)(C^{(2)} - M)$$

For simplicity of notations, we introduce a new variable $r$:

$$r \overset{\text{def}}{=} \frac{\int_\Omega \tilde{\epsilon} : \left[ \chi_1(x)(C^{(1)} - M) + \chi_2(x)(C^{(2)} - M) \right] : \epsilon \, dx}{\int_\Omega \tilde{\epsilon} : M : \epsilon \, dx}$$

Clearly, $-1 \leq r \leq 0$ for any $\epsilon \in \mathcal{E}$ because of (1)–(4). We consider the following expression:

$$\frac{\left| B(\epsilon, \epsilon) \right|}{\int_\Omega \tilde{\epsilon} : M : \epsilon \, dx} = \left| 1 + \frac{r}{s} \right|$$

Let $\text{Im}(s)$ denote the imaginary part of $s$ and $i \equiv \sqrt{-1}$. For any complex number $s = |s|e^{i\theta}$ such that $\text{Im}(s) \neq 0$, we have $|1+r/s| \geq |\sin \theta| > 0$.

For real $s$, we consider the following two cases. If $s > 1$, we have $|1+r/s| \geq \gamma$ for some constant $\gamma > 0$ because $-1 \leq r \leq 0$. If $s < 0$, $|1+r/s| \geq 1$ for the same reason.

Therefore, for $s$ outside the unit interval $[0, 1]$ on the complex plane, the sesquilinear form $B$ is coercive. Hence the existence and uniqueness of the solution $\epsilon'$ in $\mathcal{E}$ is guaranteed by the Lax–Milgram theorem. We conclude that there exists a solution $u' \in W^{1,2}_0(\Omega)$ such that

$\int_{\Omega} \epsilon' : \epsilon' \, dx$ by Korn’s inequality.

### 4. REPRESENTATION FORMULA FOR THE EFFECTIVE ELASTICITY TENSOR

The effective elasticity tensor $C^*(s)$ is defined by the relation:

$$
\mathcal{C}^0 : C^*(s) : \epsilon^0 \equiv \frac{1}{|\Omega|} \int_{\Omega} \epsilon : \Lambda(x; s) : \epsilon \, dx
$$

where $\epsilon$ solves the BVP in (6). Note that the composite we are interested in corresponds to $z = 0$, i.e. $s = 1$, hence we need the following lemma.

**Lemma 4.1**

There exists a positive number $\delta$ such that the statement in Theorem 3.1 can be extended to $s \in \mathbb{C} \setminus [0, 1 - \delta]$.

**Proof**

For $s = 1 - \delta$, we have

$$
\Lambda(x; 1 - \delta) = \chi_1(x)C^{(1)} + \chi_2(x)C^{(2)} - \delta[\chi_1(x)(M - C^{(1)}) + \chi_2(x)(M - C^{(2)})]
$$

Using (1)–(4) to obtain

$$
|B(\epsilon', \epsilon')| \geq \max\{x_1 - \delta \cdot x_{M1}, x_2 - \delta \cdot x_{M2}\} \int_{\Omega} \epsilon' : \epsilon' \, dx
$$

therefore $B$ is coercive when $\delta$ is small enough.

Define a new variable

$$
m(z) \equiv \frac{\mathcal{C}^0 : C^*(z) : \epsilon^0}{\epsilon^0 : M : \epsilon^0} \tag{10}
$$

Note that $m(1) = 1$ because for $z = 1$, the solution to BVP is $\epsilon = \epsilon^0$ due to the affine boundary condition in (6). Also, by a suitable choice of $\epsilon^0$, $m(z)$ is the ratio between $C_{ijkl}^*$ and $M_{ijkl}$ such that $i = k$ and $j = l$.

We introduce a new function

$$
F(s) \equiv 1 - m(z) \tag{11}
$$

and recall that $s \equiv 1/(1 - z)$. The coercivity of the sesquilinear form $B$ for $s \in \mathbb{C} \setminus [0, 1 - \delta]$ implies that the solution $\epsilon'$ depends analytically on the parameter $s$. Combining this with (9)–(11), we have the following proposition.
Proposition 4.1
$F(s)$ is analytic in $\mathbb{C} \setminus [0, 1 - \delta]$.

In order to obtain the integral representation formula for $F(s)$, we verify another two properties of $F(s)$.

Proposition 4.2
$\text{Im}(-F(s)) > 0$ if $\text{Im}(s) > 0$.

Proof
By definition,
$$\text{Im}(-F(s)) = \text{Im}(m(z))$$
$$= \left( \frac{\text{Im}(z)}{|\Omega|^0 : M : e^0} \right) \left\{ \int_{\Omega} \frac{\epsilon(z)}{|\Omega|^1 : \mathcal{X}_1(x)(M - C^{(1)}) + \mathcal{X}_2(x)(M - C^{(2)})} : e(z) \, dx \right\}$$
Since $\epsilon : (M - C^{(1)}) : e$, $\tilde{\epsilon} : (M - C^{(2)}) : e$, and $e^0 : M : e^0$ are all non-negative real numbers, $\text{Im}(-F)$ and $\text{Im}(z)$ must have the same sign. On the other hand,
$$\text{Im}(z) = \text{Im} \left( 1 - \frac{1}{s} \right) = \text{Im} \left( -\frac{3}{|s|^2} \right) = \text{Im} \left( \frac{s}{|s|^2} \right)$$
Hence $\text{Im}(-F(s)) > 0$ if $\text{Im}(s) > 0$. $\square$

Proposition 4.3
There exists a positive number $\mathcal{K}$ such that $|sF(is)| < \mathcal{K}$ for all $s > 0$.

Proof
Recall that $F(is) \stackrel{\text{def}}{=} 1 - m(1 - (1/is)) = 1 - m(1 + (i/s))$. Substituting $z = 1 + (i/s)$ into (5), we have
$$\Lambda \left( x; 1 + \frac{i}{s} \right) = M - \frac{i}{s}\mathcal{X}_1(x)(C^{(1)} - M) - \frac{i}{s}\mathcal{X}_2(x)(C^{(2)} - M)$$
The coercivity constant $\nu$ for the corresponding sesquilinear form $B(\epsilon', \epsilon')$ in (8) is calculated explicitly as
$$|B(\epsilon', \epsilon')| = \left| \int_{\Omega} \tilde{\epsilon} : M : \epsilon' \, dx + \frac{i}{s} \int_{\Omega} \tilde{\epsilon} : [\mathcal{X}_1(x)(M - C^{(1)}) + \mathcal{X}_2(x)(M - C^{(2)})] : \epsilon' \, dx \right|$$
$$\geq \max \left( \alpha_{M1} + \alpha_1, \frac{1}{s}(\alpha_{M1} : |\Omega_1| + \alpha_{M2} : |\Omega_2|) \right) \int_{\Omega} \tilde{\epsilon} : \epsilon' \, dx$$
i.e. $\nu = \max(\alpha_{M1} + \alpha_1, (1/s)(\alpha_{M1} : |\Omega_1| + \alpha_{M2} : |\Omega_2|))$ with $|\Omega_1|$ and $|\Omega_2|$ being the volume of $\Omega_1$ and $\Omega_2$, respectively.

The norm of the corresponding bounded linear functional $f$ in (7) is estimated using the Cauchy–Schwarz inequality:

$$|f(\epsilon')| \leq k_1 \left( \mathcal{M} + \frac{1}{s} \right) \|\epsilon'\|_{\sigma}$$

where

$$\|\epsilon'\|_{\sigma} \overset{\text{def}}{=} \left( \int_{\Omega} \left( \epsilon' : \epsilon' \right) d\mathbf{x} \right)^{1/2}$$

$$k_1 \overset{\text{def}}{=} 27|\Omega|^{1/2} \max_{k,l=1,2} \left( |\epsilon_{kl}^0| \right)$$

$$\mathcal{M} \overset{\text{def}}{=} \max_{i,j,k,l} |M_{ijkl}|$$

$$\mathcal{J} \overset{\text{def}}{=} \max_{i,j,k,l} \left( |C_{ijkl}^{(p)}| \right)$$

By Lax–Milgram theorem, we have the following estimate:

$$\|\epsilon'\|_{\sigma} \leq \|f\| = \frac{k_1 (\mathcal{M} + (1/s) \mathcal{J})}{\max(\mathcal{M}_{11} + \mathcal{M}_{12}|\Omega_1| + \mathcal{M}_{21} · |\Omega_2|)}$$  \hspace{1cm} (12)

Let $R$ be a number strictly between 0 and 1. For $s \geq 1/R$, we have $|z - 1| \leq R < 1$ because of (10). Since $m(z)$ is analytic at $z = 1$, using the Taylor expansion of $m(z)$ around $z = 1$ and the fact that $m(1) = 1$, we get

$$sF(is) = s \left[ 1 - m(1) - \sum_{k=1}^{\infty} \frac{m^{(k)}(1)}{k!} \left( \frac{i}{s} \right)^k \right] = -\sum_{k=1}^{\infty} \frac{m^{(k)}(1)}{k!} i^k \left( \frac{1}{s} \right)^k$$

Therefore,

$$|sF(is)| \leq \sum_{k=1}^{\infty} |\frac{m^{(k)}(1)}{k!}| \left( \frac{1}{s} \right)^k$$

$$\leq \sum_{k=1}^{\infty} \left| \frac{m^{(k)}(1)}{k!} \right| R^{k-1} = \frac{1}{R} \sum_{k=1}^{\infty} \left| \frac{m^{(k)}(1)}{k!} \right| R^k = \mathcal{K}_1$$

Note $\mathcal{K}_1 < \infty$ because $m(z)$ converges absolutely in $|z - 1| \leq R$.

For $s \leq \min\{1/R, (\mathcal{M}_{11} · |\Omega_1| + \mathcal{M}_{12} · |\Omega_2|)/(|\Omega_1| + \mathcal{M}_{21})\}$, we apply Young’s inequality to the definition of $F(is)$ to obtain

$$|F(is)| \leq \left[ 1 + k_2 \left( \mathcal{M} + \frac{1}{s} \mathcal{J} \right) \right] \|\epsilon'\|_{\sigma}$$  \hspace{1cm} (13)

where

$$k_2 \overset{\text{def}}{=} \frac{1}{|\Omega| |\epsilon_0^0 : \mathbf{M} : \epsilon_0^0|}$$
Substituting (12) into (13), we have

\[
|sF(is)\| \leq s \left[ 1 + k_2 \left( \mu + \frac{1}{s} \right) \right] \left( \frac{k_1(\mu + (1/s)\tilde{Z})}{(1/s)(\Omega_1 + \Omega_2)} \right)^2
\]

\[
= [s + k_2(s, \mu + \tilde{Z})] \left( \frac{k_1(s, \mu + \tilde{Z})}{\Omega_1 + \Omega_2} \right)^2
\]

\[
\leq [1/R + k_2(\mu/R + \tilde{Z})] \left( \frac{k_1(\mu/R + \tilde{Z})}{\Omega_1 + \Omega_2} \right)^2 =: \mathcal{K}_2
\]

Therefore, \(|sF(is)\| \leq \max(\mathcal{K}_1, \mathcal{K}_2)\) for all \(s > 0\).

By a general representation theorem in function theory [3,6], Propositions 4.1–4.3 imply the following theorem for \(F(s)\).

**Theorem 4.1**

There exists a positive Borel measure \(\lambda\) defined on the interval \([0, 1]\) such that for \(s\) outside \([0, 1 - \delta]\), \(F(s)\) has the following Stieltjes integral representation:

\[
F(s) = \int_0^1 \frac{\lambda(u) du}{s - u}
\]

The measure \(\lambda\) depends on the microstructure, \(e^0, C^{(1)}, C^{(2)}\) and \(M\) in (5).

**5. RELATIONS BETWEEN THE MOMENTS OF THE MEASURE \(\lambda\) AND THE MICROSTRUCTURE OF THE COMPOSITE**

The moments of \(\lambda\) in (14) are completely determined by the derivatives of \(F(s)\) at infinity, i.e. derivatives of \(m(z)\) at \(z = 1\). In order to prove this statement, we expand the right-hand side of (14) near \(s = \infty\) (i.e. near \(z = 1\)) to obtain

\[
F(s) = \frac{1}{s} \int_0^1 \frac{\lambda(u) du}{1 - u/s} = \frac{1}{s} \int_0^1 \sum_{k=0}^{\infty} \left( \frac{u}{s} \right)^k \lambda(u) du = \frac{1}{s} \int_0^1 \sum_{k=0}^{\infty} \frac{1}{s^{k+1}} \int_0^1 u^k \lambda(u) du
\]

The last equality is due to the uniform convergence of \(\sum_{k=0}^{\infty} \left( u/s \right)^k\).

On the other hand, using the definition of \(F(s)\) in (11), the Taylor expansion of \(F(s)\) near \(s = \infty\) is

\[
F(s) \equiv 1 - m(z) = 1 - \left[ m(1) + \sum_{k=1}^{\infty} \frac{m^{(k)}(1)}{k!} (z - 1)^k \right] = \sum_{k=0}^{\infty} \frac{m^{(k+1)}(1)}{(k + 1)!} \frac{(-1)^k}{s^{k+1}}
\]

Equating the coefficients of \(s^{-k}\) leads to the following relations:

\[
(-1)^k \frac{m^{(k+1)}(1)}{(k + 1)!} = \int_0^1 u^k \lambda(u) du \equiv \mu^{(k)}(\lambda), \quad k = 0, 1, 2, \ldots
\]

Therefore, the \(k\)th moment \(\mu^{(k)}\) of the positive measure \(\lambda\) is determined by the \(k + 1\) derivative of \(m(z)\) at \(z = 1\).
Next, we derive an explicit relation between \( m^{(k)}(1) \) and the microstructure of the composite, hence the relations between the microstructure and the moments of \( \lambda \).

For \( k = 1 \), differentiating (10) leads to

\[
m'(z) = \frac{1}{|\Omega| e^0 : M : e} \left\{ \int_{\Omega} \tau_z : \Lambda : \epsilon \, dx + \int_{\Omega} \tau : \Lambda : \epsilon_z \, dx + \int_{\Omega} \tau : \Lambda_z : \epsilon \, dx \right\}
\]

(16)

where the subscript \( z \) denotes partial derivative with respect to \( z \).

The first term on the right-hand side of (16) equals 0 because \( u_z \) (the derivative of \( u \) with respect to \( z \)) vanishes on the boundary \( \partial \Omega \) and \( u \) satisfies BVP as well as natural transmission conditions on the interface \( \mathcal{I} \) between \( \Omega_1 \) and \( \Omega_2 \), i.e.

\[
\sum_{i,j,k,l=1}^{3} \int_{\Omega} (\tau_z)_{ij} \Lambda_{ijkl} \epsilon_{kl} \, dx = \sum_{i,j,k,l=1}^{3} \int_{\Omega} \frac{\partial (\tau_z)}{\partial x_j} \Lambda_{ijkl} \frac{\partial u_k}{\partial x_l} \, dx
\]

\[
= \sum_{i,j,k,l=1}^{3} \left[ \int_{\Omega \cap \mathcal{I}} (\tau_z)_{ij} \Lambda_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \, dS - \int_{\Omega} \frac{\partial (\tau_z)}{\partial x_j} \left( \Lambda_{ijkl} \frac{\partial u_k}{\partial x_l} \right) \, dx \right]
\]

\[
= 0
\]

where the first equality is due to the symmetry \( \Lambda_{ijkl} = \Lambda_{klij} \) for \( i,j,k,l = 1 \ldots 3 \). The second term on the right-hand side of (16) vanishes for the same reason. Therefore, we have

\[
m'(1) = \frac{1}{|\Omega| e^0 : M : e^0} \int_{\Omega} \tau_{|z=1} : [\chi_1(M - C^{(1)}) + \chi_2(M - C^{(2)})] : e_{|z=1} \, dx
\]

(17)

Note that \( e_{|z=1} \) equals to the constant tensor \( e^0 \) specified in the affine boundary condition because \( \Lambda(1, x) = M \). Thus, \( m'(1) \) depends neither on the microstructure nor on the geometry of \( \Omega \) and it is

\[
m'(1) = \frac{e^0 : [M - C^{(2)} + p_1(C^{(2)} - C^{(1)})]}{e^0 : M : e^0}
\]

(17)

where \( p_1 \overset{\text{def}}{=} |\Omega_1|/|\Omega| \) is the volume fraction of \( \Omega_1 \).

For \( k \geq 2 \), we have

\[
m^{(k)}(z) = \frac{1}{|\Omega| e^0 : M : e^0} \int_{\Omega} \sum_{i,j,p,s=1}^{3} [\chi_1(M - C^{(1)}) + \chi_2(M - C^{(2)})]_{ijps} \frac{\partial^{k-1}}{\partial z^{k-1}} (\tau_{ij} e_{ps}) \, dx
\]

Therefore, the evaluation of \( m^{(k)}(1), k \geq 2 \), requires knowledge of the derivatives \( \frac{\partial^n}{\partial z^n} \tau_{ij} |_{z=1} \) for \( n = 1 \) up to \( n = k - 1 \). Since \( \Lambda(1, x) = M \), this can be done by using the Green’s tensor function \( g \) [7], which satisfies the following set of equations:

\[
\begin{align*}
\sum_{i,j,l=1}^{3} M_{ijkl} \frac{\partial^2}{\partial x_l \partial x_j} g_{ml}(x, x') + \delta_{mk} \delta(x - x') = 0, & \quad m, k = 1 \ldots 3 \\
g_{ml}(x, x') = 0 & \quad \text{for } x' \in \partial \Omega,
\end{align*}
\]

together with the set of equations satisfied by \( \frac{\partial \varepsilon}{\partial z^n} \bigg|_{z=1} \), \( n \geq 1 \),

\[
\begin{cases}
    \varepsilon = \frac{1}{2} (\nabla u + \nabla^T u) \\
    \text{Div} \left( M : \frac{\partial \varepsilon}{\partial z^n} \bigg|_{z=1} \right) = -n \text{Div} \left( \Lambda_z : \frac{\partial^{n-1} \varepsilon}{\partial z^{n-1}} \bigg|_{z=1} \right) \\
    \left( \frac{\partial^n u}{\partial z^n} \right)_{z=1} = 0
\end{cases}
\]

\( n \geq 1 \) with \( \varepsilon|_{z=1} = \varepsilon^0 \). Therefore for \( k \geq 2 \), \( m^{(k)}(1) \) and \( \mu^{(k-1)} \) depend on the \( k \)-point correlation function of \( \Omega_1 \) and \( \Omega_2 \) because \( \Lambda_z = \chi_1(M - C^{(1)}) + \chi_2(M - C^{(2)}) \).

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