On the representation formula for well-ordered elastic composites: 
A convergence of measure approach

Miao-Jung Ou∗, †

Department of Mathematics, University of Central Florida, Orlando, FL 32816, U.S.A.

Communicated by R. P. Gilbert

SUMMARY

The aim of this paper is to derive an integral representation formula for the effective elasticity tensor for a two-component well-ordered composite of elastic materials without using a third reference medium and without assuming the completeness of the eigenspace of the operator \( \hat{G} \) defined in (2.16) in (J. Mech. Phys. Solids 1984; 32(1):41–62). As shown in (J. Mech. Phys. Solids 1984; 32(1):41–62) and (Math. Meth. Appl. Sci. 2006; 29(6):655–664), this integral representation formula implies a relation which links the effective elastic moduli to the \( N \)-point correlation functions of the microstructure. Such relation not only facilitates a powerful scheme for systematic incorporation of microstructural information into bounds on the effective elastic moduli but also provides a theoretical foundation for de-homogenization. The analysis presented in this paper can be generalized to an \( n \)-component composite of elastic materials. The relations developed here can be applied to the de-homogenization for a special class of linear viscoelastic composites. The results will be presented in another paper. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: integral representation formula; well-ordered composites; microstructure; positive Borel measure; Helly’s theorems

1. INTRODUCTION

The integral representation formula was derived in the course of systematic evaluation of bounds on the effective dielectric tensor [1] and on the effective elasticity tensor of composites [2] via analytic continuation method. Unlike the variational method approach used in the seminal paper by Hashin and Shtrikman [3], this approach does not rely on the availability of variational principles. Due to the availability of the integral representation formula, the effective permittivity/elasticity tensor can be regarded as a linear functional on the set of positive finite Borel measures on interval \([0, 1] \) with prescribed moments which contain the microstructural information of the composite.

∗Correspondence to: Miao-Jung Ou, Department of Mathematics, University of Central Florida, Orlando, FL 32816, U.S.A.
†E-mail: mou@mail.ucf.edu

Received 31 August 2006
The estimation of bounds is then formulated as an optimization problem over the set of Borel measures with constrained moments, see [1, 2]. Bounds of all orders, including classical bounds such as the Voigt bound, the Reuss bound and the Hashin–Shtrikman bound, can be derived using this approach.

In [1], the integral representation formula for the effective dielectric tensor of a mixture of two isotropic materials was derived in the Hilbert space setting assuming random stationary fields. In [2], the derivation was based on the assumption that the eigenspace of the operator \( \hat{G} \) defined in Equation (2.16) [2] is complete. The validity of this assumption is not easy to check upon because \( \hat{G} \) depends also on the microstructure. The derivation of the integral representation formula for effective tensor for composites of two arbitrary elastic materials without this assumption is given in [4] in a Hilbert space setting, with the help of a third reference medium whose elasticity tensor is denoted by \( \mathbf{M} \) in (1). The derivation of the integral representation formula in the Hilbert space setting for elastic composites with holes or hard inclusions can be found in [5].

The importance of deriving the integral representation formula for effective elasticity tensor for well-ordered elastic composites without using a third reference medium and without assuming completeness of the eigenspace is twofold. As mentioned in [2], for a well-ordered composite with a complete eigenspace, the Hashin–Shtrikman bounds can be derived by using this representation formula. Also, the de-homogenization scheme in [6] for recovering volume fractions from measurement of effective elasticity tensors of different frequencies is based on the representation formula derived in this paper.

We organize this paper as follows. In Section 2, the notations and symbols used in this paper are defined and explained. To make this paper self-contained, we give a brief summary of the results from [4] in Section 3. The main theorem is then stated and proved in Section 4.

2. NOTATIONS AND SYMBOLS

All the boldface upper case Roman letters denote symmetric fourth-order tensors, whereas the boldface lower case Roman letters denote vectors. The boldface lower case Greek letters denote symmetric matrices. The transpose of matrix \( \epsilon \) is denoted by \( \epsilon^\top \).

A fourth-order tensor \( \mathbf{C} \) is symmetric if its components, denoted by \( C_{ijkl}, i, j, k, l = 1 \ldots 3 \), have the following properties:

\[
C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}
\]

Tensor contraction between \( \epsilon^{(1)} \) and \( \epsilon^{(2)} \) is defined as

\[
\epsilon^{(1)} : \epsilon^{(2)} \overset{def}{=} \sum_{i,j=1}^{3} \epsilon_{ij}^{(1)} \epsilon_{ij}^{(2)}
\]

The \( ij \)-component of the right contraction product between fourth-order tensor \( \mathbf{C} \) and matrix \( \epsilon \) is defined as

\[
[\mathbf{C} : \epsilon]_{ij} \overset{def}{=} \sum_{k,l=1}^{3} C_{ijkl} \epsilon_{kl}
\]
Similarly, the $ij$-component of the left contraction product between fourth-order tensor $\mathbf{C}$ and matrix $\mathbf{\epsilon}$ is defined as

$$\{\mathbf{\epsilon} : \mathbf{C}\}_{ij} \overset{\text{def}}{=} \sum_{k,l=1}^{3} \epsilon_{kl} C_{klij}$$

Tensor $\mathbf{C}$ is greater than tensor $\mathbf{A}$, denoted by $\mathbf{C} \succ \mathbf{A}$ if

$$\mathbf{\bar{\epsilon}} : \mathbf{C} : \mathbf{\epsilon} \succ \mathbf{\bar{\epsilon}} : \mathbf{A} : \mathbf{\epsilon} \quad \forall \text{ symmetric matrix } \mathbf{\epsilon}$$

where $\mathbf{\bar{\epsilon}}$ is the complex conjugate of $\mathbf{\epsilon}$. The infinity-norm of $\mathbf{A}$ is defined as

$$\|\mathbf{A}\|_\infty \overset{\text{def}}{=} \max_{i,j,k,l=1}^{3} |A_{ijkl}|$$

The gradient of a vector function $\mathbf{u}$ is defined as

$$\left(\nabla \mathbf{u}\right)_{ij} \overset{\text{def}}{=} \frac{\partial u_i}{\partial x_j}$$

The divergence operator $\text{Div}$ with respect to $\mathbf{x} \in \mathbb{R}^3$ acting on a matrix $\mathbf{\epsilon}$ is defined as

$$\left(\text{Div} \mathbf{\epsilon}\right)_{k} \overset{\text{def}}{=} \sum_{i=1}^{3} \frac{\partial \epsilon_{ki}}{\partial x_i}, \quad k = 1 \ldots 3$$

The region occupied by materials I and II are denoted by $\Omega_1, \Omega_2$, respectively. As usual, $\chi_i$ is the characteristic function for region $\Omega_i$, $i = 1, 2$. $\Omega \overset{\text{def}}{=} \Omega_1 \cup \Omega_2$ with its measure denoted by $|\Omega|$. The space $\mathcal{E}_0$ and $\mathcal{E}$ are defined as

$$\mathcal{E}_0 \overset{\text{def}}{=} \left\{ \mathbf{\epsilon}(\mathbf{x}) \mid \mathbf{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \mathbf{u} \in (C_0^1(\Omega))^3 \right\}$$

$$\mathcal{E} \overset{\text{def}}{=} \text{closure of } \mathcal{E}_0 \text{ in } (L^2(\Omega))^9$$

$$\|\mathbf{\epsilon}\|_{(L^2(\Omega))^9} \overset{\text{def}}{=} \sqrt{\int_{\Omega} \mathbf{\epsilon} : \mathbf{\epsilon} \, d\mathbf{x}} = \sqrt{\sum_{i,j=1}^{3} \|\epsilon_{ij}\|_{L^2(\Omega)}^2}$$

3. REPRESENTATION FORMULA FOR EFFECTIVE ELASTIC MODULI OF TWO-COMPONENT COMPOSITE WITH A THIRD REFERENCE MATERIAL

In [4], it is shown that for the family of composites with the elasticity tensors parameterized by $z$ in the following form:

$$\Lambda(\mathbf{x}; z) = \mathbf{M} + (1 - z)\chi_1(\mathbf{x})(\mathbf{C}(1) - \mathbf{M}) + (1 - z)\chi_2(\mathbf{x})(\mathbf{C}(2) - \mathbf{M}) \quad (1)$$

Copyright © 2006 John Wiley & Sons, Ltd.  

DOI: 10.1002/mma
the effective elasticity tensor $C^*(z)$ is analytic and has the following \textit{integral representation formula} for every $z$ outside the negative real axis on the complex plane

$$ \overline{e}^0 : C^*(z) : e^0 = \overline{e}^0 : M : e^0 \left(1 - \int_0^1 \frac{\lambda \, du}{s - u}\right) $$ (2)

where the new variable $s$ is defined as $s \overset{\text{def}}{=} 1/(1 - z)$ and $\lambda(du)$ is a positive Borel measure independent of $z$. Here, the effective elasticity tensor $C^*$ is defined by the relation

$$ \overline{e}^0 : C^*(s) : e^0 \overset{\text{def}}{=} \frac{1}{|\Omega|} \int_\Omega \tilde{e} : \Lambda(x; s) : \epsilon \, dx $$ (3)

with $\epsilon$ being the solution of the boundary value problem (BVP) which reads

$$ \begin{cases} 
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), & i, j = 1 \ldots 3 \\
(\text{Div}(\Lambda(x; z) : \epsilon))_i = 0, & i = 1 \ldots 3 \\
u_i|_{\partial \Omega} = \sum_{j=1}^3 \epsilon_{ij} x_j, & i = 1 \ldots 3
\end{cases} $$ (4)

with the transmission conditions of continuity of the displacement $u$ and the normal stress across the interface between $\Omega_1$ and $\Omega_2$. Here, $e^0$ is a constant symmetric tensor. It is shown in [4] that (BVP) has a unique solution for every $z$ in $C \setminus (-\infty, 0)$. In order for the representation formula to hold true, the following restrictions on $C^{(1)}$, $C^{(2)}$ and the reference elasticity tensor $M$ needs to be imposed for all symmetric second-order tensors $\epsilon$

$$ \begin{align*} 
z_1 \tilde{\epsilon} : \epsilon &\leq \tilde{\epsilon} : C^{(1)} : \epsilon \leq \beta_1 \tilde{\epsilon} : \epsilon \\
z_2 \tilde{\epsilon} : \epsilon &\leq \tilde{\epsilon} : C^{(2)} : \epsilon \leq \beta_2 \tilde{\epsilon} : \epsilon \\
z_{M1} \tilde{\epsilon} : \epsilon &\leq \tilde{\epsilon} : (M - C^{(1)}) : \epsilon \leq \beta_{M1} \tilde{\epsilon} : \epsilon \\
z_{M2} \tilde{\epsilon} : \epsilon &\leq \tilde{\epsilon} : (M - C^{(2)}) : \epsilon \leq \beta_{M2} \tilde{\epsilon} : \epsilon
\end{align*} $$ (5-8)

for some positive constants $z_1, z_2, \beta_1, \beta_2, z_{M1}, \beta_{M1}, \beta_{M2}$.

Note that the constant second-order tensor $e^0$ plays the role of selecting components of $C^*$. For example, for the choice of $\epsilon_{ij}^0 = \delta_{i1} \delta_{j2}$, we have $\overline{e}^0 : C^*(z) : e^0 = \overline{e}^{1212}$. In this way, integral representation formulas for different components of $C^*$ can be derived separately.

4. REPRESENTATION FORMULA FOR EFFECTIVE ELASTIC MODULI OF WELL-ORDERED TWO-COMPONENT COMPOSITE WITHOUT A THIRD REFERENCE MATERIAL

As can be seen in Proposition 4.3 of [4], the proof of existence of the representation formula (2) relies on the assumption that $z_{M1} > 0$ and $z_{M2} > 0$ in (7) and (8) so that $(1/s) \cdot \min(z_{M1}, z_{M2}) > z_{M1} + z_1$.
when $s$ is small enough.\(^4\) Clearly, this assumption fails when $M$ is set to be equal to the larger tensor $C^{(1)}$ because $\min(z_{M1}, z_{M2}) = 0$ for this case. In this section, we show that the integral representation formula (2) indeed can be adapted to the case $M = C^{(1)} \geq C^{(2)}$. We would like to mention that the integral representation formula for the case of $M = C^{(1)} \geq C^{(2)}$ was derived by Kantor and Bergman [2] for the purpose of deriving bounds on elastic moduli of two-component composites. Their derivation is under the assumption that the operator $\hat{G}$ defined in Equation (2.16) in [2] has a complete eigenspace which can be applied to decompose the identity operator. The validity of this assumption is not easy to check upon because the microstructure of the composite is present in the definition of $\hat{G}$. Our proof does not depend on this assumption.

The main theorem of this paper is stated as follows.

**Theorem 4.1**

Let $A_\infty(x, s) \defeq C^{(1)} + \frac{1}{s} \chi_s(x)(C^{(2)} - C^{(1)})$, $C^{(1)} \geq C^{(2)}$. Consider the boundary value problem

$$(BV P^-) \quad \begin{cases} \epsilon_\infty = \frac{1}{2}(\nabla u_\infty + \nabla^T u_\infty) \\ \text{Div}(A_\infty(s) : \epsilon_\infty) = 0 \quad \text{in } \Omega \\ u_\infty = \epsilon^0 x \quad \text{on } \partial \Omega \end{cases}$$

(a) There exist a unique solution of $(BV P^-)$ for $s \in \mathbb{C} \setminus [0, 1]$.

(b) The effective elasticity tensor function

$$\epsilon^0 : C^{(1)}_\infty(s) : \epsilon^0 \defeq \frac{1}{|\Omega|} \int_\Omega \epsilon_\infty : A_\infty(x; s) : \epsilon_\infty \, dx$$

is analytic outside [0, 1] on the $s$-plane.

(c) There exists a positive Borel measure $\lambda_\infty$ on [0, 1] such that

$$\epsilon^0 : C^{(1)}_\infty(z) : \epsilon^0 = \epsilon^0 : C^{(1)} : \epsilon^0 \left(1 - \int_0^1 \frac{\lambda_\infty du}{s - u}\right)$$

for $s \in \mathbb{C} \setminus [0, 1]$.

**Proof (For Theorem 4.1(a) and (b))**

We consider the variational formulation of $(BV P^-)$.

Find $\epsilon' \in \mathcal{E}$ such that

$$\int_\Omega \tilde{\epsilon} : A_\infty : (\epsilon^0 + \epsilon') \, dx = 0 \quad \forall \epsilon \in \mathcal{E}$$

(9)

\(^4\)We would like to correct the typographical errors in [4] here. First, all the terms $z_{M1} \cdot |\Omega_1| + z_{M2} \cdot |\Omega_2|$ should be changed to $\min(z_{M1}, z_{M2})$. Secondly, the term $\|\epsilon^0\|_2$ should be added to $\|\epsilon\|_2$ in Equation (13) and the estimate of $X_2$. Thirdly, a factor of 2 should be multiplied to the estimate of $X_2$. Finally, the estimate of $|B(\epsilon', \epsilon')|$ should be $\min(x_1 - \delta/(1 - \delta) \cdot \beta_{M1}, x_2 - \delta/(1 - \delta) \cdot \beta_{M2})$. Nevertheless, the validity of the proof in [4] is not affected by these typographical errors.
where \( \bar{e} \) denotes the complex conjugate of \( e \). Note the sesquilinear form

\[
B_\infty(e, e') \overset{\text{def}}{=} \int_{\Omega} \bar{e} : A_\infty(x; s) : e' \, dx
\]

is bounded in the Hilbert space \( \mathcal{E} \times \mathcal{E} \) for any fixed \( s \). For simplicity of notations, we introduce a new variable \( r \)

\[
r \overset{\text{def}}{=} \int_{\Omega} \bar{e} : [Z_2(x)(C^{(2)} - C^{(1)})] : e' \, dx
\]

\[
= \frac{1}{\int_{\Omega} \bar{e} : C^{(1)} : e' \, dx}
\]

Clearly, \(-1 \leq r \leq 0\) for any \( e \in \mathcal{E} \) because of (6) and the assumption \( C^{(1)} \geq C^{(2)} \). To show the coercivity of \( B_\infty \), we consider the following expression:

\[
\frac{|B_\infty(e, e)|}{\int_{\Omega} \bar{e} : C^{(1)} : e' \, dx} = \left| 1 + \frac{r}{s} \right|
\]

Let \( \text{Im}(s) \) denote the imaginary part of \( s \) and \( i \overset{\text{def}}{=} \sqrt{-1} \). For any complex number \( s = |s|e^{i\theta} \) such that \( \text{Im}(s) \neq 0 \), we have \(|1 + r/s| \geq |\sin \theta| > 0\).

For real \( s \), we consider the following two cases. If \( s > 1 \), we have \(|1 + r/s| \geq \gamma \) for some constant \( \gamma > 0 \) because \(-1 \leq r \leq 0\). If \( s < 0 \), \(|1 + r/s| \geq 1 \) for the same reason.

Therefore, for \( s \) outside the unit interval \([0, 1]\) on the complex plane, the sesquilinear form \( B_\infty \) is coercive. Hence, the existence and uniqueness of the solution \( e' \) in \( \mathcal{E} \) is guaranteed by the Lax–Milgram theorem. We conclude that there exists a solution \( u' \in W^{1,2}_0(\Omega) \) such that \( e' = \frac{1}{2} (\nabla u' + \nabla^T u') \) by Korn’s inequality. The analyticity of \( F(s) \) is a consequence of the coercivity of the sesquilinear form \( B_\infty \).

The proof for Theorem 4.1(c) is built on the results mentioned in the previous section by using a convergence of measure argument. We first construct a sequence of positive symmetric fourth-order tensors which converges strongly to \( C^{(1)} \) from above, i.e. we consider \( \{M_n\} \) such that

\[
\|M_n - C^{(1)}\|_\infty \to 0 \quad \text{as} \ n \to \infty
\]

\[
\xi_n \ll \bar{e} : (M_n - C^{(1)}) : e \ll \zeta_n \bar{e} : e \quad \forall n \quad \text{and} \quad \xi_n \searrow 0, \quad \zeta_n \searrow 0 \quad \text{as} \ n \to \infty
\]

For example, we can choose \( \{M_n\} = \{(1 + \delta_n)C^{(1)}\} \) with \( \delta_n \searrow 0 \). Since each \( M_n \) in the sequence satisfies conditions (7) and (8), each element of this sequence corresponds to a unique solution and a positive Borel measures \( \lambda_n \) on \([0, 1]\), according to the results given in (2). The major part of our proof for Theorem 4.1(c) is to show that the sequence of measures \( \{\lambda_n\} \) weak-star converges to a positive Borel measure on \([0, 1]\) and to show that this limit measure indeed provides the desired integral representation formula for the effective elasticity tensor \( C^*_\infty(\cdot) \) defined in Theorem 4.1(b).

We first prove the following lemmas.
Lemma 4.1
For every fixed $s$ in $\mathbb{C} \setminus [0, 1]$, let $\epsilon_n(s)$ be the unique solution of the boundary value problem
\begin{align}
(\text{BVP-aux}) \quad 
\begin{cases}
\epsilon_{ij} &= \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), & i, j = 1 \ldots 3 \\
(\text{Div}(A_n(x; s) : \epsilon)) &= 0, & i = 1 \ldots 3 \\
u_i|_{\partial \Omega} &= \sum_{j=1}^{3} \epsilon_{ij} x_j, & i = 1 \ldots 3
\end{cases}
\end{align}
where $A_n(s) = \chi_1 [M_n + (1/s)(C^{(1)} - M_n)] + \chi_2 [M_n + (1/s)(C^{(2)} - M_n)]$ for $M_n$ defined in (12).
Then the sequence $\{\epsilon_n(s)\}$ is bounded in $(L^2(\Omega))^9$.

Proof
The variational formulation of (BVP-aux) reads
\begin{align}
\int_{\Omega} \bar{\epsilon} : A_n : (\epsilon^0 + \epsilon') \, dx = 0 \quad \forall \epsilon \in \mathcal{E}
\end{align}
where $\bar{\epsilon}$ denotes the complex conjugate of $\epsilon$. Similar to the estimate of (11), the sesquilinear form
\begin{align}
B_n(\epsilon, \epsilon') \overset{\text{def}}{=} \int_{\Omega} \bar{\epsilon} : A_n(x; s) : \epsilon' \, dx
\end{align}
can be shown to satisfy the following coercive condition for some positive number $\gamma(s) \in \mathbb{C} \setminus [0, 1]$
\begin{align}
|B_n(\epsilon', \epsilon')| \geq \gamma(s) (\chi_1 + \bar{\chi}_n) \int_{\Omega} \bar{\epsilon} : \epsilon' \, dx \quad \forall \epsilon', \epsilon' \in \mathcal{E}
\end{align}
where $\chi_1$ and $\bar{\chi}_n$ are defined by $C^{(1)}$ and $M_n$ as in (5) and (12), respectively. On the other hand, the linear functional $f_n(\epsilon') \overset{\text{def}}{=} \int_{\Omega} \bar{\epsilon} : A_n : \epsilon^0 \, dx$ satisfies the following estimate:
\begin{align}
|f_n(\epsilon')| \leq 9 \max \left( \left\| M_n + \frac{1}{s} (C^{(1)} - M_n) \right\|_{\infty}, \left\| M_n + \frac{1}{s} (C^{(2)} - M_n) \right\|_{\infty} \right) \left\| \epsilon' \right\|_{L^2(\Omega)} \left\| \epsilon^0 \right\|_{(L^2(\Omega))^9}
\end{align}
Since $\left\| M_n - C^{(1)} \right\|_{\infty} \to 0$ as $n \to \infty$, the estimate above implies that there exists a positive finite constant $k(s)$ such that
\begin{align}
|f_n(\epsilon')| \leq k(s) \left\| \epsilon' \right\|_{L^2(\Omega)} \left\| \epsilon^0 \right\|_{(L^2(\Omega))^9}
\end{align}
Therefore, the solution $\epsilon'_n$ to the variational formulation satisfies the estimate
\begin{align}
\left\| \epsilon'_n \right\|_{\mathcal{E}} \leq \frac{k(s) \left\| \epsilon^0 \right\|_{L^2(\Omega)} \left\| \epsilon^0 \right\|_{(L^2(\Omega))^9}}{\gamma(s) (\chi_1 + \bar{\chi}_n)} \leq \frac{k(s) \left\| \epsilon^0 \right\|_{L^2(\Omega)} \left\| \epsilon^0 \right\|_{(L^2(\Omega))^9}}{\bar{\gamma}(s) \cdot \bar{\chi}_n}
\end{align}
Hence, the $(L^2(\Omega))^9$-norm of the solution to (BVP-aux) $\left\| \epsilon_n \right\|_{L^2(\Omega)} = \left\| \epsilon'_n + \epsilon^0 \right\|_{L^2(\Omega)}$ is bounded as well.
Next, the solution \( \epsilon_n(s) \) and the sesquilinear form \( B_n(\cdot, \cdot) \) in (15) are used to define the effective elasticity tensor \( C^*_n(s) \) and the auxiliary function \( m_n(s) \)

\[
\overline{\epsilon}^0 : C^*_n(s) : \epsilon^0 \overset{\text{def}}{=} \frac{B_n(\epsilon_n(s), \epsilon_n(s))}{|\Omega|}
\]

\[
m_n(s) \overset{\text{def}}{=} \frac{\overline{\epsilon}^0 : C^*_n(s) : \epsilon^0}{\overline{\epsilon}^0 : M_n : \epsilon^0}
\]

We define \( m_\infty(s) \) in a similar fashion by using the solution of problem (BVP-\( \infty \))

\[
m_\infty(s) \overset{\text{def}}{=} \frac{\overline{\epsilon}^0 : C^*_\infty(s) : \epsilon^0}{\overline{\epsilon}^0 : C(1) : \epsilon^0} = \int_\Omega \overline{\epsilon}^\infty : \Lambda_\infty(x; s) : \epsilon_\infty \, dx
\]

**Lemma 4.2**

For every \( s \in C \setminus [0, 1] \), there exists a subsequence of \( \{m_n(s)\} \) which converges to \( m_\infty(s) \).

**Proof**

By the weak compactness of \( L^2(\Omega) \) (Banach–Alaoglu theorem), Lemma 4.1 implies that for every \( s \in C \setminus [0, 1] \), there exists a subsequence \( \{\epsilon'_n(s)\} \) such that

\[
\epsilon'_n(s) \wsto \epsilon'_\infty(s) \quad \text{for some } \epsilon'_\infty(s) \in (L^2(\Omega))^9 \quad \text{as } n_k \to \infty
\]

Consider the variational formulation of \( \epsilon'_n(s) \) in (BVP-aux)

\[
\int_\Omega \overline{\epsilon} : \Lambda_n(s) : (\epsilon'_n + \epsilon^0) \, dx = 0 \quad \forall \epsilon \in \mathcal{E}
\]

Passing the limits \( \Lambda_n(s) \to \Lambda_\infty \) and \( \epsilon'_n \wsto \epsilon'_\infty \) in (16) leads to

\[
\int_\Omega \overline{\epsilon} : \Lambda_\infty(s) : (\epsilon'_\infty + \epsilon^0) \, dx = 0 \quad \forall \epsilon \in \mathcal{E}
\]

which coincides with the variational formulation of (BVP-\( \infty \)) for \( \epsilon'_\infty(14) \). By the uniqueness of the solution, we conclude

\[
\epsilon'_n(s) \wsto \epsilon'_\infty(s) \quad \text{as } n_k \to \infty
\]

Note \( m_{n_k}(s) \) can be written as

\[
m_{n_k}(s) = \frac{B_n(\epsilon'_n(s) + \epsilon^0, \epsilon'_n(s) + \epsilon^0)}{|\Omega| : \overline{\epsilon}^0 : M_n : \epsilon^0}
\]

Furthermore, (16) implies

\[
B_n(\epsilon'_n(s) + \epsilon^0, \epsilon'_n(s) + \epsilon^0) = B_n(\epsilon^0, \epsilon'_n(s) + \epsilon^0)
\]

Substituting (19) into (18) and letting \( n_k \to \infty \), we conclude that \( m_{n_k}(s) \to m_\infty(s) \) as \( n_k \to \infty \) because of (17) and the fact \( M_{n_k} \to C(1) \). \( \square \)
For clarity of notations, the following lemma is stated in terms of the variable $z$ rather than $s$. Recall that $s \overset{\text{def}}{=} 1/(1-z)$ and $m_n$ is analytic for $s \in \mathbb{C} \setminus [0, 1)$, i.e., for $z \in \mathbb{C} \setminus (-\infty, 0)$.

**Lemma 4.3**
The derivative $m'_{nk}(z = 1)$ tends to $m'_{\infty}(z = 1) < \infty$ as $n_k \to \infty$.

**Proof**
The coercivity of all the sesquilinear form $B_{nk}$ and $B_{\infty}$ implies that $\epsilon'_{nk}$ and $\epsilon_{\infty}$ are analytic functions of $z$ with values in $(L^2(\Omega))^q$. Therefore, we may differentiate (18) on both sides with respect to $z$ to obtain

$$m'_{nk}(z) = \int_\Omega \nabla_z (\epsilon_{nk} \Lambda_{nk} : \nabla (\epsilon_{nk} + \epsilon_0)) \, dx + \int_\Omega \nabla_z (\epsilon_{nk} \Lambda_{nk} : (\epsilon_{nk} + \epsilon_0)) \, dx + \int_\Omega \nabla_z (\epsilon_{nk} \Lambda_{nk} : \nabla x (\epsilon_{nk} + \epsilon_0)) \, dx$$

As shown in [4] through integration by parts, these derivatives at $z = 1$ can be simplified to

$$m'_{nk}(1) = \frac{\overline{\epsilon_0} : \left[M_{nk} - C^{(2)} + p_1(C^{(2)} - C^{(1)})\right]}{\overline{\epsilon_0} : M_{nk} : \epsilon_0}$$

Similarly,

$$m'_{\infty}(1) = \frac{\overline{\epsilon_0} : \left[C^{(1)} - C^{(2)} + p_1(C^{(2)} - C^{(1)})\right]}{\overline{\epsilon_0} : C^{(1)} : \epsilon_0}.$$ 

Therefore, $m'_{nk}(1) \to m'_{\infty}(1)$ because $M_{nk} \to C^{(1)}$ as $n_k \to \infty$. \hfill \Box

In order to prove Theorem 4.1(c), we shall also need Helly’s theorems, which can be found in [7].

**Theorem 4.2 (Helly’s first theorem)**
Let $\{f_n\}$ be a sequence of functions on $[a, b]$ such that the variation of $f_n$ on $[a, b]$ is uniformly bounded by $k$. Then there exists a subsequence $\{f_{nk}\}$ which converges pointwise on $[a, b]$ to a function $f$ whose variation on $[a, b]$ is also bounded by $k$.

**Theorem 4.3 (Helly’s second theorem)**
Suppose $\{f_n\}$ is a sequence of functions with bounded variations on $[a, b]$. If $f_n \to f$ pointwise on $[a, b]$ and the variations of $f_n$ is uniformly bounded, then $f$ has bounded variation on $[a, b]$ and

$$\lim_{n \to \infty} \int_a^b g \, df_n = \int_a^b g \, df \quad \text{for every continuous function } g \text{ on } [a, b]$$

**Proof (For Theorem 4.1(c))**
It is shown in [2, 4] that the moments of the positive Borel measure $\hat{\lambda}_n$ are related to the function $m_{nk}$ and its derivatives at $z = 1$ by the relations

$$(-1)^q \frac{m_n(q+1)}{(q+1)!} = \int_0^1 u^q \hat{\lambda}_n(u) \, du \overset{\text{def}}{=} \mu^{(q)}(\hat{\lambda}_n), \quad q = 0, 1, 2, \ldots \quad (20)$$
Let $\sigma_n$ denote the normalized non-decreasing distribution function of measure $\lambda_n$ such that $\sigma_n(0) = 0$ and $\sigma_n$ is left-continuous, see [8, p. 223]. Since $\int_0^1 d\sigma_n$ is exactly the total variation of $\sigma_n$ on $[0, 1]$ and (20) implies $m_n'(1) = \int_0^1 d\sigma_n$, we may apply Lemma 2 to conclude that the sequence of functions $\{\sigma_n\}$ is uniformly bounded in the closed interval $[0, 1]$. By Theorem 4.2, there exists a subsequence $\{\sigma_{n_k}\}$ which converges weakly to a non-decreasing, left-continuous function on $[0, 1]$. We denote this function by $\sigma_\infty$. It is non-decreasing and left continuous because it is the limit of functions with these properties. Applying Theorem 4.3, we have

$$\lim_{n_k \to \infty} \int_0^1 \frac{1}{s-u} d\sigma_{n_k}(u) = \int_0^1 \frac{1}{s-u} d\sigma_\infty(u) \quad \text{for } \forall s \in C \setminus [0, 1]$$

(21)

Furthermore, the representation formula (2) and the definition of $m_{n_k}(s)$ imply

$$\int_0^1 \frac{1}{s-u} d\sigma_{n_k}(u) = 1 - m_{n_k}(s)$$

(22)

Therefore, the sequence $\{1 - m_{n_k}(s)\}$ strongly converges to $\int_0^1 (1/(s-u)) d\sigma_\infty(u)$ for every $s$ in $C \setminus [0, 1]$. On the other hand, Lemma 4.2 implies that for every $s \in C \setminus [0, 1]$, there exists a subsequence of $\{1 - m_{n_k}(s)\}$ which converges to $1 - m_\infty(s)$. The uniqueness of limit leads to the conclusion

$$1 - m_\infty(s) = \int_0^1 \frac{1}{s-u} d\sigma_\infty(u)$$

Since $\sigma_\infty$ is a non-decreasing function of bounded variation on $[0, 1]$, its derivative exists almost everywhere. This proves the existence of positive measure $\lambda_\infty$ such that

$$\int_0^1 \frac{1}{s-u} d\sigma_\infty(u) = \int_0^1 \frac{\lambda_\infty}{s-u} du$$

REFERENCES