Borrower: DLM

Lending String: *JHE,KKU,IPL,MFM,FDA

Patron: Ou, Miao-Jung

Journal Title: Fifth International Conference on Mathematical and Numerical Aspects of Wave Propagation /

Volume: Issue: 2000
Pages: 482-488

Article Author:

Article Title: Implementation of The Method of Variation of Boundaries for Three-dimensional Objects in a Waveguide


ILL Number: 99214826

Call #: QC157.i58 2000 c. 1

Location:

Borrower: DLM
1/16/2013 1:05:30 PM

Mail/Ariel
Charge
Maxcost: 40.00FM

Shipping Address:
University of Delaware Library - Interlibrary Loan
181 South College Avenue
Newark, DE 19717-5267

Fax: [Number]
Ariel: 128.175.82.31
Email: ILL@hawkins.lib.udel.edu
ODYSSEY IP: 206.107.43.75
Implementation of the method of variation of boundaries for three-dimensional objects in a waveguide

J. L. Buchanan†  R. P. Gilbert‡  M. Ou‡

Abstract

The Method of Variation of Boundaries MVB[2, 3, 4, 5] has been used in space for determining the shape of an unknown object from measured values of the acoustic field generated by scattering of an incident wave by the object. In this paper we use this method for a fully three-dimensional object in a wave guide. In previous papers we have used MVB to generate the field and the Intersecting Canonical Body Approach ICBA [6], [9] to construct the object. The main effort for developing an algorithm consists of constructing recurrence relations for the coefficients in the representation of the scattered field.

1 The unidentified object problem

We investigate an inverse problem associated with a shallow ocean. With regard to acoustic waves the ocean acts as a guide of uniform depth, having a pressure release surface and a completely reflecting bottom. Suppose that the ocean occupies the region \( \Omega := \mathbb{R}^2 \times [0, h] \), the surface being designated by \( \Omega_0 := \mathbb{R}^2 \times \{0\} \), and the bottom by \( \Omega_h := \mathbb{R}^2 \times \{h\} \). In several previous works [6] [9] we developed the Intersecting Canonical Domain Body Approximation ICBA for determining the shape of an unknown scatterer situated in the water column. The MVB as well as the ICBA take advantage of the fact that for any cylinder \( D_d \) having axis perpendicular to the wave guide surfaces, the total wave field may be represented in \( \Omega - D_d \) by

\[
  u(r, \theta, z) = u^{inc}(r, \theta, z) + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \phi_n(z) H_n^{(1)}(kr \alpha_n r)e^{i\theta - \theta_0},
\]

where the incident "plane wave" is

\[
  u^{inc}(r, \theta, z) = \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} i^m \phi_n(z) \phi_n(z_0) J_n(kr \alpha_n r)e^{i\theta - \theta_0},
\]

and

\[
  N = \left\lfloor \frac{1}{2} + \frac{kh}{\pi} \right\rfloor
\]

*This research was supported in part by the Naval Research Laboratory through grant National Science Foundation through grants BES-9402539 and INT-9726213.

†Department of Mathematics, U. S. Naval Academy, Annapolis, MD 21402, U.S.A.

‡Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, U.S.A.
\[ \phi_n(z) = \sqrt{\frac{2}{h}} \sin \left( \frac{(2n-1)\pi z}{2h} \right) \]
\[ a_n = \sqrt{1 - \frac{(n - \frac{1}{2})^2 \pi^2}{k^2 h^2}}. \]

Here the \( \phi_n(z) \) are modal solutions. The functions \( J_m \) and \( H_m^{(1)} \) are the \( m^{th} \) order Bessel and Hankel functions of the first kind, respectively. The scattered solution has the form

\[ u^{sc}(r, \theta, z) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \phi_n(z) H_m^{(1)}(k a_n r) e^{im(\theta - \theta_0)}. \]

For the case in which the object is a sound-soft circular cylinder coaxial with the \( z \)-axis and having radius \( R \), the coefficients \( A_{nm} \) are given by, [6]

\[ A_{nm} = -i^m \phi_n(z_0) \frac{J_m(k a_n R)}{H_m^{(1)}(k a_n R)} \]

2 Derivation of the recurrences for calculation of the scattered field

The method of variation of boundaries, MVB, [2, 3, 4, 5] is applicable when the scatterer may be considered as a small perturbation of a shape for which the direct scattering problem is explicitly solvable. In the case of an ocean of uniform depth a suitable shape is a circular cylinder for which the solution to the scattering problem is well known [6]. Indeed, in [1] this method to generate the acoustic field for depth independent cylinders having a cross section of the form \( r = R + \delta f(\theta) \). In the present work we consider fully three-dimensional objects which may be described in terms of a shape function form \( r = R + \delta f(\theta, z) \). To develop a MVB algorithm we expand the scattered wave in terms of the perturbation parameter \( \delta \)

\[ (3) \quad u^{sc}(r, \theta, z) = \sum_{l=0}^{\infty} \delta^l u_l(r, \theta, z) \]

where each \( u_l \) is assumed to have the form of the canonical solution

\[ u_l(r, \theta, z) = \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} b_{nm,l} \phi_n(z) H_m^{(1)}(k a_n r) e^{im(\theta - \theta_0)} \]

The unknown coefficients \( b_{nm,l} \) are determined by the condition imposed at the surface of the scatterer. At the surface \( r = R + \delta f(\theta, z) \) of the obstacle, assuming a sound-soft boundary condition,

\[ (4) \quad u^{inc}(R + \delta f(\theta, z), \theta, z) = -\sum_{p=0}^{\infty} \delta^p u_p(R + \delta f(\theta, z), \theta, z). \]

As only \( N \) modes propagate we shall only concern ourselves with these modes; hence, by setting \( \delta = 0 \) we obtain from

\[ \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} i^m \phi_n(z) \phi_n(z_0) J_m(k a_n R) e^{im(\theta - \theta_0)} \]
\[ = -\sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} b_{nm;0} \phi_n(z) H_m^{(1)}(k a_n R) e^{im(\theta - \theta_0)} \]
the first expansion coefficients

\[ b_{nm;0} = - \frac{\imath^m}{H_m^{(1)}(ka_n R)} \phi_n(z_0) J_m(ka_n R) \]

The higher-order coefficients are obtained by differentiating (4) \( p \) times to get

\[ \frac{d^p}{d\delta^p} u^{\text{inc}} = - \sum_{\ell=0}^{\infty} \sum_{q=0}^{p-1} \binom{p}{q} \frac{d^q}{d\delta^q} \delta^\ell \frac{d^{p-q}}{d\delta^{p-q}} u_\ell. \]

Upon setting \( \delta = 0 \) we get

\[ \frac{d^p}{d\delta^p} u^{\text{inc}} \bigg|_{\delta=0} = - \sum_{\ell=0}^{\infty} \sum_{q=0}^{p-1} \binom{p}{q} \frac{d^q}{d\delta^q} \delta^\ell \frac{d^{p-q}}{d\delta^{p-q}} u_\ell \bigg|_{\delta=0} = - \sum_{\ell=0}^{\infty} \sum_{q=0}^{p-1} \binom{p}{q} \frac{\ell!}{(\ell-q)!} \delta^{\ell-q} \frac{d^{p-q}}{d\delta^{p-q}} u_\ell \bigg|_{\delta=0} \]

whence

\[ u_p \bigg|_{\delta=0} = - \frac{1}{p!} \frac{d^p}{d\delta^p} u^{\text{inc}} \bigg|_{\delta=0} = \sum_{\ell=0}^{p-1} \frac{1}{(p-\ell)!} \frac{d^{p-\ell}}{d\delta^{p-\ell}} u_\ell \bigg|_{\delta=0}. \]

The first term on the right hand side of (6) is

\[ - \frac{1}{p!} \frac{d^p}{d\delta^p} u^{\text{inc}} \bigg|_{\delta=0} = \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} \imath^m \phi_n(z_0) \phi_n(z_0) J_{nm;p} \cdot (f(\theta, z))^p e^{i m(\theta - \theta_0)} \]

where

\[ J_{nm;p} = (ka_n)^p \frac{d^p}{d\tau^p} J_m^{(p)}(\tau) \bigg|_{\tau=ka_n R}. \]

Denoting the Fourier expansion of \( f(\theta, z)^p \) in terms of the modal solutions \( \phi_s(z) \) by

\[ f(\theta, z)^p = \sum_{q=-\infty}^{\infty} \sum_{s=1}^{N} \Gamma_{p,s,q} \phi_s(z) e^{i q(\theta - \theta_0)} \]

yields

\[ - \frac{1}{p!} \frac{d^p}{d\delta^p} u^{\text{inc}} \bigg|_{\delta=0} = \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{s=1}^{N} \Gamma_{p,s,q} \phi_n(z_0) \phi_n(z_0) J_{nm;p} \cdot \phi_s(z) e^{i [m+q](\theta - \theta_0)}. \]

Since

\[ \phi_n(z)\phi_s(z) = \frac{2}{h} \sin \left( \frac{(2n-1)\pi z}{2h} \right) \sin \left( \frac{(2s-1)\pi z}{2h} \right) \]

\[ = \frac{1}{h} \left( \cos \left( \frac{(n-s)\pi z}{h} \right) - \cos \left( \frac{(n+s-1)\pi z}{h} \right) \right) \]

\[ = \frac{1}{\sqrt{h}} \left( \frac{1}{\sqrt{\varepsilon_{n-s}}} \psi_{n-s}(z) - \frac{1}{\sqrt{\varepsilon_{n+s-1}}} \psi_{n+s-1}(z) \right) \]
where
\[ \psi_n(z) := \sqrt{\frac{\varepsilon_n}{h}} \cos \left( \frac{n\pi z}{h} \right), \]
we have
\[ -\frac{1}{p!} \left. \frac{d^p}{d\delta^p} u^{inc}_{\delta=0} \right|_{\delta=0} \]
\[ = - \sum_{n=1}^{N} \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i^m e^{i(m+q)(\theta-\theta_0)} \cdot \phi_n(z_0) J_{nm;p} \]
\[ \cdot \left( \sum_{s=1}^{n} \Gamma_{p,s,q} \frac{1}{\sqrt{h\varepsilon_{n-s}}} \psi_{n-s}(z) + \sum_{s=n+1}^{N} \Gamma_{p,s,q} \frac{1}{\sqrt{h\varepsilon_{s-n}}} \psi_{s-n}(z) - \sum_{s=1}^{N} \Gamma_{p,s,q} \frac{1}{\sqrt{h\varepsilon_{s+n-1}}} \psi_{s+n-1}(z) \right) \]
The second term on the right hand side of (6) is
\[ -\sum_{\ell=0}^{p-1} \frac{1}{(p-\ell)!} \frac{d^\ell}{d\delta^\ell} u_{\delta=0} \]
\[ = - \sum_{\ell=0}^{p-1} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{N} \sum_{q=-\infty}^{\infty} \phi_n(z) \left( \frac{k a_n}{(p-\ell)!} \right) \left( \frac{H_1^{(1)}}{H_m^{(1)}} \right)^{p-\ell} \frac{1}{(p-\ell)!} \cdot f(\theta, z)^{p-\ell} b_{nm;\ell} e^{i m(\theta-\theta_0)} \]
\[ = - \sum_{\ell=0}^{p-1} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{N} \sum_{q=-\infty}^{\infty} b_{nm;\ell} H_{nm;p-\ell} \cdot e^{i(m+q)(\theta-\theta_0)} \cdot \Gamma_{p-\ell,s,q} \phi_n(z) \phi_s(z) \]
\[ = - \sum_{\ell=0}^{p-1} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{N} \sum_{q=-\infty}^{\infty} b_{nm;\ell} H_{nm;\ell} e^{i(m+q)(\theta-\theta_0)} \]
\[ \cdot \left( \sum_{s=1}^{n} \Gamma_{p-\ell,s,q} \frac{1}{\sqrt{h\varepsilon_{n-s}}} \psi_{n-s}(z) + \sum_{s=n+1}^{N} \Gamma_{p-\ell,s,q} \frac{1}{\sqrt{h\varepsilon_{s-n}}} \psi_{s-n}(z) \right) \]
\[ - \sum_{s=1}^{N} \Gamma_{p-\ell,s,q} \frac{1}{\sqrt{h\varepsilon_{s+n-1}}} \psi_{s+n-1}(z) \]
where
\[ H_{nm;\ell} \equiv \left( \frac{k a_n}{(p-\ell)!} \frac{d^\ell}{d\tau^\ell} H_1^{(1)}(\tau) \right|_{\tau=k a_n R}. \]

Multiplying the expanded version of (6) by \( \phi_j(z) \) and integrating from 0 to \( h \) gives
\[ \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} b_{nm;\ell} H_m^{(1)} \int_0^h \phi_n(z) \phi_j(z) dz \cdot e^{i m(\theta-\theta_0)} \]
\[ = - \sum_{n=1}^{N} \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i^m e^{i(m+q)(\theta-\theta_0)} \phi_n(z_0) J_{nm;p} \left[ \sum_{s=1}^{n} \Gamma_{p,s,q} \frac{1}{\sqrt{h\varepsilon_{n-s}}} \int_0^h \psi_{n-s}(z) \phi_j(z) dz \right] \]
\[ + \sum_{s=n+1}^{N} \Gamma_{p,s,q} \frac{1}{\sqrt{h\varepsilon_{s-n}}} \int_0^h \psi_{s-n}(z) \phi_j(z) dz - \sum_{s=1}^{N} \Gamma_{p,s,q} \frac{1}{\sqrt{h\varepsilon_{s+n-1}}} \int_0^h \psi_{s+n-1}(z) \phi_j(z) \]
Using the orthonormality of the \( \phi_n(z) \) and the identity

\[
I_{nm} := \int_0^h \phi_n(z)\psi_m(z) \, dz = 2\sqrt{2\varepsilon_m} \frac{2n - 1}{\pi (2n - 1)^2 - 4m^2}
\]

gives

\[
\begin{aligned}
&\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} b_{nm;\ell} H_{nm;\ell} e^{im(\theta - \theta_0)} \\
= &\sum_{n=1}^{N} \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} i^{m_q} e^{i(m+q)(\theta - \theta_0)} \phi_n(z_0) J_{nm;\ell} \psi_m(z) \\
&\left[ \sum_{s=1}^{n} \frac{1}{\sqrt{h\varepsilon_{n-s}}} \frac{1}{\sqrt{h\varepsilon_{s-n}}} \right] \\
&+ \sum_{s=n+1}^{N} \frac{1}{\sqrt{h\varepsilon_{s-n}}} \frac{1}{\sqrt{h\varepsilon_{s-n}}} I_{j,s-n} \\
&- \sum_{s=1}^{N} \frac{1}{\sqrt{h\varepsilon_{s-n}}} \frac{1}{\sqrt{h\varepsilon_{s-n}}} I_{j,s+n-1}
\end{aligned}
\]

Multiplying both sides of the equation above by \( e^{im'(\theta - \theta_0)} \) and integrating from 0 to 2\( \pi \) gives

\[
\begin{aligned}
&H_{nm';\ell}^{(1)} (ka_j R) b_{jm';\ell} \\
= &\sum_{n=1}^{N} \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i^{m_q} \phi_n(z_0) J_{n,m'-q,p} \psi_m(z) \\
&\left[ \sum_{s=1}^{n} \frac{1}{\sqrt{h\varepsilon_{n-s}}} \frac{1}{\sqrt{h\varepsilon_{s-n}}} I_{j,s-n} \\
&- \sum_{s=1}^{N} \frac{1}{\sqrt{h\varepsilon_{s-n}}} \frac{1}{\sqrt{h\varepsilon_{s-n}}} I_{j,s+n-1}
\end{aligned}
\]

### 3 Calculation of the Bessel derivative and Fourier series coefficients

Following [?], we use the identity

\[
\frac{d^l}{dr^l} J_m(r) = \frac{1}{2^l} \sum_{p=0}^{l} (-1)^p \binom{l}{p} J_{m-l+2p}(r)
\]

to compute the coefficients \( J_{nm;l} \) and \( H_{nm;l} \).
Recursive relations for the coefficients of the Fourier series of

\[ f(\theta, z) = \sum_{q=-\infty}^{\infty} \sum_{s=1}^{N} \Gamma_{s,q} \phi_s(z) e^{i q (\theta - \theta_0)} \]

may be obtained in terms of those of the Fourier series

\[ f(\theta, z) = \sum_{q=-\infty}^{\infty} \sum_{s=1}^{N} \Delta_{s,q} \psi_s(z) e^{i q (\theta - \theta_0)} \]

by starting with

\[ \Gamma_{\ell,s,q} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{h} f(\theta, z) e^{i \ell (\theta - \theta_0)} d\theta d \theta' \]

\[ \Delta_{\ell,s,q} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{h} f(\theta, z) e^{i \ell (\theta - \theta_0)} d\theta d \theta' \]

and observing that from (8)

\[ \Gamma_{\ell,s,q} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{h} f(\theta, z) e^{i \ell (\theta - \theta_0)} \left( \sum_{p=1}^{N} \sum_{k=-\infty}^{\infty} \Gamma_{1,p,k} \phi_p(z) e^{i k (\theta - \theta_0)} \right) \phi_s(z) e^{-i q (\theta - \theta_0)} d\theta d z \]

\[ = \frac{1}{2\pi} \sum_{p=1}^{N} \sum_{k=-\infty}^{\infty} \Gamma_{1,p,k} \int_{0}^{2\pi} \int_{0}^{h} f(\theta, z) e^{i \ell (\theta - \theta_0)} \phi_p(z) \phi_s(z) e^{-i (p-k) (\theta - \theta_0)} d\theta d z \]

\[ = \frac{1}{2\pi} \left[ \sum_{p=1}^{N} \sum_{k=-\infty}^{\infty} \Gamma_{1,p,k} \int_{0}^{2\pi} \int_{0}^{h} f(\theta, z) e^{i \ell (\theta - \theta_0)} \phi_p(z) \phi_s(z) e^{-i (p-k) (\theta - \theta_0)} d\theta d z \right. \]

\[ + \sum_{p=s+1}^{N} \sum_{k=-\infty}^{\infty} \Gamma_{1,p,k} \int_{0}^{2\pi} \int_{0}^{h} f(\theta, z) e^{i \ell (\theta - \theta_0)} \phi_p(z) \phi_s(z) e^{-i (p-k) (\theta - \theta_0)} d\theta d z \]

\[ - \sum_{p=1}^{N} \sum_{k=-\infty}^{\infty} \Gamma_{1,p,k} \int_{0}^{2\pi} \int_{0}^{h} f(\theta, z) e^{i \ell (\theta - \theta_0)} \phi_p(z) \phi_s(z) e^{-i (p-k) (\theta - \theta_0)} d\theta d z \]

\[ = \sum_{p=1}^{N} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{\epsilon_{p,s-k}}} \Gamma_{1,p,k} \Delta_{\ell-1,s-k,q-k} + \sum_{p=s+1}^{N} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{\epsilon_{p,s-k}}} \Gamma_{1,p,k} \Delta_{\ell-1,p-s-q-k} \]

Similarly since

\[ \psi_p(z) \psi_q(z) = \frac{\sqrt{\epsilon_p}}{h} \cos \left( \frac{p \pi z}{h} \right) \sqrt{\epsilon_q} \cos \left( \frac{q \pi z}{h} \right) \]

\[ = \frac{1}{2} \sqrt{\epsilon_p} \sqrt{\epsilon_q} \left( \cos \left( \frac{(p-q) \pi z}{h} \right) + \cos \left( \frac{(p+q) \pi z}{h} \right) \right) \]

\[ = \frac{1}{2} \sqrt{\epsilon_p} \sqrt{\epsilon_q} \left( \sqrt{\frac{h}{\epsilon_p-q}} \psi_{p-q} + \sqrt{\frac{h}{\epsilon_p+q}} \psi_{p+q} \right) \]
we have

\[
\Delta_{\ell,s,q} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(\theta, z)^{\ell-1} \left( \sum_{p=1}^N \sum_{k=-\infty}^\infty \Delta_{1,p,k} \psi_p(z) e^{ik(\theta-\theta_0)} \right) \psi_s(z) dz e^{-i(q-k)(\theta-\theta_0)} d\theta
\]

\[
= \frac{1}{2\pi} \sum_{p=1}^N \sum_{k=-\infty}^\infty \Delta_{1,p,k} \int_0^{2\pi} \int_0^{2\pi} f(\theta, z)^{\ell-1} \psi_p(z) \psi_s(z) e^{-i(q-k)(\theta-\theta_0)} d\theta dz
\]

\[
= \frac{1}{2\pi} \left[ \sum_{p=1}^N \sum_{s=-\infty}^\infty \Delta_{1,p,k} \int_0^{2\pi} \int_0^{2\pi} e^{-i(q-k)(\theta-\theta_0)} f(\theta, z)^{\ell-1} \frac{1}{2} \sqrt{\frac{\epsilon_p \epsilon_s}{h \epsilon_{s-p}}} \psi_{s-p}(z) d\theta dz
\]

\[
+ \sum_{p=s+1}^\infty \Delta_{1,p,k} \int_0^{2\pi} \int_0^{2\pi} e^{-i(q-k)(\theta-\theta_0)} f(\theta, z)^{\ell-1} \frac{1}{2} \sqrt{\frac{\epsilon_p \epsilon_s}{h \epsilon_{p-s}}} \psi_{p-s}(z) d\theta dz
\]

\[
= \sum_{p=1}^N \sum_{k=-\infty}^\infty \frac{1}{2} \sqrt{\frac{\epsilon_p \epsilon_s}{h \epsilon_{s-p}}} \Delta_{1,p,k} \Delta_{\ell-1,s-p,q-k}
\]

\[
+ \sum_{p=s+1}^\infty \sum_{k=-\infty}^\infty \frac{1}{2} \sqrt{\frac{\epsilon_p \epsilon_s}{h \epsilon_{p-s}}} \Delta_{1,p,k} \Delta_{\ell-1,p-s,q-k}
\]

References


