LAMB WAVES IN A POROELASTIC PLATE

ROBERT P. GILBERT
Department of Mathematical Sciences
University of Delaware
Newark DE 19716, USA

DOO-SUNG LEE
Department of Mathematics
College of Education, Konkuk University
1 Hwayang-Dong, Kwangjin-Gu, Seoul, Korea

M. YVONNE OU
Department of Mathematical Sciences
University of Delaware
Newark DE 19716, USA

Received 18 November 2011
Revised
Accepted 7 December 2012

Keywords:

1. Introduction

Currently, bone mineral density (BMD) is the gold standard for \textit{in vivo} assessment of fracture risk of bones and is measured using X-ray absorptiometric techniques.\textsuperscript{1} However, only 70–80\% of the variance of bone strength is accounted for by bone density. As the brittleness of bone depends on more factors than bone density, biologists believe that quantitative ultrasound techniques (QUT) could provide an important new diagnostic tool.\textsuperscript{2–4} Moreover, in contrast to X-ray densitometry, ultrasound does not ionize the tissue, and its implementation is relatively inexpensive. Since the loss of bone density and the destruction of the bone microstructure is most evident in osteoporosis cancellous bone, it is natural to consider
the possibility of developing accurate ultrasound models for the sonification of cancellous bone. It would be of enormous clinical advantage if an accurate method could be developed using ultrasound interrogation to determine whether one had osteoporosis. The intention of this research is to eventually produce an accurate clinical procedure for determining the bone density and other bone parameters describing bone brittleness.

Some researchers\textsuperscript{5–9} have sought to use the Biot formulation of a poroelastic material to determine bone architecture. The Biot theory which is obtained from mixture theory can only be applied in the low frequency range (\(< 100\, \text{kHz}\)), which corresponds to wave lengths sufficiently larger than the pore size; see also Ref. \textsuperscript{10} for a homogenization approach. The resonance range would be about 500 Khz and the ultrasound range of 1–2 MHz is in the scattering range. Luppé et al have studied the effect of multiple scattering in Refs. \textsuperscript{11–13}.

Biot’s theory predicts a fast and slow compressional wave. The second compressional wave does not exist for elastic material, so the detection of two different compressional waves signifies the poroelastic property of a specimen. Hosokawa and Otoni\textsuperscript{14} (see also McKelvie and Palmer)\textsuperscript{9} identified fast and slow waves in cancellous bone. In this paper, we consider the special case where the surface of the plate is unloaded, which coincides with Lamb waves in the elastic case. Physically this situation is not practical as the fluid, unless it was very viscous, would then leak out because of the pore pressure. The problem we solve here is a mathematically interesting toy problem. Physically speaking the more reasonable problem would be to consider there to be fluid pressure on both sides of the plate. This problem can be solved using the solution we derive here and will be presented in a subsequent work.

2. Biot Equations

Cortical and cancellous bones are the two main types of the osseous tissue, and they may be considered to be poroelastic in structure. We consider a Biot model of a poroelastic material treats it as an elastic frame with interstitial pore fluid. In the Biot theory\textsuperscript{7} the displacement vectors $u(x, y, z, t)$ and $v(x, y, z, t)$ track the motions of the frame and fluid, respectively in a three-dimensional material. Also we use the notation $e = \nabla \cdot u$ and $\epsilon = \nabla \cdot v$ to denote the frame and fluid dilatations, respectively. For expository reasons we explain our procedure to two-dimensional problems. For this case the following constitutive relations are assumed

\begin{align*}
\sigma_{xx} &= 2\mu e_{xx} + \lambda e + Qe, \\
\sigma_{yy} &= 2\mu e_{yy} + \lambda e + Qe, \\
\sigma_{zz} &= 2\mu e_{zz} + \lambda e + Qe, \\
\sigma_{xy} &= \mu e_{xy}, \\
\sigma_{xz} &= \mu e_{xz}, \\
\sigma_{yz} &= \mu e_{yz}, \\
\sigma &= Qe + Re.
\end{align*}

Here $\lambda$ and $\mu$ are the Lamé coefficients of the elastic frame, $R$ is a parameter measuring the pressure on the fluid required to force a certain volume of fluid into the sediment at
Table 1. Parameters in the Biot model.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_f$</td>
<td>Density of the pore fluid</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>Density of frame material</td>
</tr>
<tr>
<td>$K_b$</td>
<td>Complex frame bulk modulus</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Complex frame shear modulus</td>
</tr>
<tr>
<td>$K_f$</td>
<td>Fluid bulk modulus</td>
</tr>
<tr>
<td>$K_r$</td>
<td>Frame material bulk modulus</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Porosity</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Viscosity of pore fluid</td>
</tr>
<tr>
<td>$k$</td>
<td>Permeability</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Structure constant</td>
</tr>
<tr>
<td>$a$</td>
<td>Pore size parameter</td>
</tr>
</tbody>
</table>

constant volume, and $Q$ measures the coupling of changes in the volume of the solid and fluid. As usual the strains are related to the displacements by

$$
e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y},$$

(2)

The parameter $\mu$, the frame shear modulus is measured. The other parameters $\lambda, R$ and $Q$ occurring in the constitutive equations are calculated from the measured or estimated values of the parameters given in Table 1 using the formulas

$$
\lambda = K_b - \frac{2}{3} \mu + \frac{(K_r - K_b)^2 - 2\beta K_r (K_r - K_b) + \beta^2 K_r^2}{D - K_b},
$$

(3)

$$
R = \frac{\beta^2 K_r^2}{D - K_b},
$$

$$
Q = \frac{\beta K_r ((1 - \beta)K_r - K_b)}{D - K_b},
$$

where

$$
D = K_r (1 + \beta (K_r/K_f - 1)).
$$

(4)

In the frequency domain, the bulk and shear moduli $K_b$ and $\mu$ are often given imaginary parts to account for frame inelasticity. We will consider the time-harmonic case here, where $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})e^{i\omega t}$, etc, time-harmonic Eqs. (1) and (2) and an argument based upon Lagrangian dynamics are shown to lead to the following equations of motion for the displacements and dilatations given in Refs. 15–18, and in particular as

$$
(\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \wedge (\nabla \wedge \mathbf{u}) + Q\nabla(\nabla \cdot \mathbf{v})
$$

$$
+ \omega^2 (\rho_{11} \mathbf{u} + \rho_{12} \mathbf{v}) - i\omega (b(\omega)(\mathbf{u} - \mathbf{v})) = 0,
$$

(5)

$$
Q\nabla(\nabla \cdot \mathbf{u}) + R\nabla(\nabla \cdot \mathbf{v}) + \omega^2 (\rho_{12} \mathbf{u} + \rho_{22} \mathbf{v}) + i\omega (b(\omega)(\mathbf{u} - \mathbf{v})) = 0
$$

(6)
\[
\rho_{11} = (1 - \beta)\rho_r + \rho_f T \beta - \beta \rho_f,
\]
\[
\rho_{12} = \beta \rho_f (1 - T),
\]
\[
\rho_{22} = \rho_f T \beta,
\]
where \(T\) is the tortuosity. These equations can be put in the more compact form:
\[
(\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) + Q \nabla (\nabla \cdot \mathbf{v}) + \omega^2 (p_{11} \mathbf{u} + p_{12} \mathbf{v}) = 0,
\]
\[
Q \nabla (\nabla \cdot \mathbf{u}) + R \nabla (\nabla \cdot \mathbf{v}) + \omega^2 (p_{12} \mathbf{u} + p_{22} \mathbf{v}) = 0,
\]
where
\[
p_{11} = \rho_{11} - \frac{b(\omega)}{\omega},
\]
\[
p_{12} = \rho_{12} + \frac{b(\omega)}{\omega},
\]
\[
p_{22} = \rho_{22} - \frac{b(\omega)}{\omega}.
\]
Using an improvement over the Biot–Stoll model, made by Johnson\(^{20}\) to replace the Biot assumption of circular-cylindrical pores by a more accurate dissipation term, Fellah \textit{et al.}\(^{3,21,22}\) replace the standard Biot equations with a Johnson-type dissipation. In the time-harmonic case these become
\[
\mu \Delta \mathbf{u} + \nabla ((\lambda + \mu) \nabla \cdot \mathbf{u} + Q \nabla \cdot \mathbf{v}) = -\omega^2 (\tilde{\rho}_{11} \mathbf{u} + \tilde{\rho}_{12} \mathbf{v}),
\]
\[
\nabla (Q \nabla \cdot \mathbf{u} + R \nabla \cdot \mathbf{v}) = -\omega^2 (\tilde{\rho}_{12} \mathbf{u} + \tilde{\rho}_{22} \mathbf{v}).
\]
Here \(\tilde{\rho}_{11}, \tilde{\rho}_{12}, \tilde{\rho}_{22}\) correspond to the mass coupling terms in the Biot’s model. They are defined in terms of solid density \(\tilde{\rho}_s\), pore fluid density \(\tilde{\rho}_f\), \(\beta\), \(\alpha_\infty\), the tortuosity which is a function of frequency \(\omega\), as
\[
\tilde{\rho}_{11} := (1 - \beta)\tilde{\rho}_s + \beta \tilde{\rho}_f (\alpha(\omega) - 1),
\]
\[
\tilde{\rho}_{12} := -\beta \tilde{\rho}_f (\alpha(\omega) - 1),
\]
\[
\tilde{\rho}_{22} := \beta \tilde{\rho}_f (\alpha(\omega) - 1).
\]
The coupling between the fluid part (marrow) and elastic matrix (trabecular bone) is described by the Johnson–Koplik–Dashen model.\(^{20}\) In this model, the dynamic tortuosity \(\alpha(\omega)\) is expressed as a function of tortuosity \(\alpha_\infty\), pore fluid viscosity \(\eta\), pore fluid density \(\tilde{\rho}_f\), permeability \(k\), porosity \(\beta\), the angular frequency \(\omega\) and the viscous characteristic length \(\Lambda\)
\[
\alpha(\omega) = \alpha_\infty \left(1 + \frac{i\eta \beta}{\omega \alpha_\infty \tilde{\rho}_f k} \sqrt{1 + \frac{4\alpha_\infty^2 k^2 \tilde{\rho}_f \omega}{i\eta \Lambda^2 \beta^2}}\right),
\]
\[i = \sqrt{-1}.
\]
Lamb Waves in a Poroelastic Plate

If Biot’s model is used, we have

\[ \tilde{\rho}_{11} := \rho_{11} + \frac{ib}{\omega}, \quad \tilde{\rho}_{22} := \rho_{22} + \frac{ib}{\omega}, \quad \tilde{\rho}_{12} := \rho_{12} - \frac{ib}{\omega}, \] (12)

where \( \tilde{b} \) is the Fourier transform of the dissipation kernel \( b(t) \). A decomposition of the Biot equations is suggested in Ref. 23 where two scalar potentials and a vector potential are used, namely

\[ u = \nabla \phi_f + \nabla \phi_s + \nabla \wedge \vec{\psi}, \quad \text{where} \quad \nabla \cdot \vec{\psi} = 0, \]

\[ v = m_1 \nabla \phi_f + m_2 \nabla \phi_s + m_3 \nabla \wedge \vec{\psi}, \quad \text{where} \quad \nabla \cdot \vec{\psi} = 0. \]

This leads to a coupled system for the fast and slow potentials, namely

\[
(\lambda + 2\mu) \Delta^2 (\phi_f + \phi_s) + \omega^2 \tilde{\rho}_{11} \Delta (\phi_f + \phi_s) + Q \Delta^2 (m_1 \phi_f + m_2 \phi_s)
+ \omega^2 \tilde{\rho}_{12} \Delta (m_1 \phi_f + m_2 \phi_s) - iwb \Delta ([1 - m_1] \phi_f + [1 - m_2] \phi_s) = 0, \\
Q \Delta^2 (\phi_f + \phi_s) + R \Delta^2 (m_1 \phi_f + m_2 \phi_s) + \omega^2 \tilde{\rho}_{12} \Delta (\phi_f + \phi_s)
+ \omega^2 \tilde{\rho}_{22} \Delta (m_1 \phi_f + m_2 \phi_s) - iwb \Delta (\phi_f + \phi_s) + iwb \Delta (m_1 \phi_f + m_2 \phi_s) = 0,
\]

which may be expanded as,

\[
(\lambda + 2\mu + m_1 Q) \Delta^2 \phi_f + (\omega^2 (\tilde{\rho}_{11} + \tilde{\rho}_{12} m_1) - iwb(1 - m_1)) \Delta \phi_f
+ (\lambda + 2\mu + m_2 Q) \Delta^2 \phi_s + (\omega^2 (\tilde{\rho}_{11} + \tilde{\rho}_{12} m_2) - iwb(1 - m_2)) \Delta \phi_s = 0 \quad (13)
\]

and

\[
(Q + m_1 R) \Delta^2 \phi_f + (\omega^2 (\tilde{\rho}_{12} + \tilde{\rho}_{22} m_1) - iwb(1 - m_1)) \Delta \phi_f
+ (Q + m_2 R) \Delta^2 \phi_s + (\omega^2 (\tilde{\rho}_{12} + \tilde{\rho}_{22} m_2) - iwb(1 - A_s)) \Delta \phi_s = 0. \quad (14)
\]

These equations uncouple to form a sixth-order system for the fast and slow wave potentials.

Following the scheme used for elastic waves, we may use the decomposition into scalar and vector potentials to describe the type of waves we have. If \( \nabla \times (\phi_f + \phi_s) = 0 \), then \( u = \nabla \times \vec{\psi} \) and we have a shear wave. On the other hand, if \( \nabla \cdot u = 0 \) this implies that

\[ \nabla \cdot \mathbf{u} = \nabla \cdot \nabla \times (\phi_f + \phi_s) \]

and so

\[ \nabla \times \mathbf{u} = \nabla \times \nabla \times (\phi_f + \phi_s) = 0; \]

hence, we have a compression wave. Let us note, that in terms of the components of the vector and scalar potentials, there are five unknowns, namely \( \phi_f, \phi_s, \psi_x, \psi_y, \psi_z \). We can write out the three displacement components as

\[
\begin{align*}
u_x &= \phi_{f,x} + \phi_{s,x} + \psi_{z,y} - \psi_{y,z}, \\
u_y &= \phi_{f,y} + \phi_{s,y} + \psi_{x,z} - \psi_{z,x}, \\
u_z &= \phi_{f,z} + \phi_{s,z} + \psi_{y,x} - \psi_{x,y}.
\end{align*}
\]
As in the elastic case we have a decomposition into compression waves, horizontal shear waves and vertical shear waves. The table below indicates how this is done. Recall that there are two compression waves, a fast one and a slow one.

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-Wave</td>
<td>$u = \nabla \phi_f + \nabla \phi_s$</td>
<td>$\nabla \wedge u = \vec{0}$</td>
</tr>
<tr>
<td></td>
<td>$u_z \neq 0, \nabla \cdot u \neq 0$</td>
<td>$\nabla \cdot u \neq 0$</td>
</tr>
<tr>
<td></td>
<td>$\nabla \wedge u = 0, (\nabla \wedge u)_z = 0$</td>
<td></td>
</tr>
<tr>
<td>SV Wave</td>
<td>$\phi_f = \phi_s = 0$</td>
<td>$u = \nabla \wedge (0, 0, \psi)$</td>
</tr>
<tr>
<td></td>
<td>$u_z \neq 0, \nabla \cdot u = 0$</td>
<td>$\nabla \wedge u = 0$</td>
</tr>
<tr>
<td></td>
<td>$\nabla \wedge u \neq 0, (\nabla \wedge u)_z = 0$</td>
<td></td>
</tr>
<tr>
<td>SH Wave</td>
<td>$\phi_f + \phi_s = 0$</td>
<td>$u = \nabla \wedge (0, 0, \chi)$</td>
</tr>
<tr>
<td></td>
<td>$u_z = 0, \nabla \cdot u = 0$</td>
<td>$\nabla \wedge u \neq 0, (\nabla \wedge u)_z \neq 0$</td>
</tr>
<tr>
<td></td>
<td>$\nabla \cdot u = 0$</td>
<td>$u_z = 0, \nabla \cdot u = 0$</td>
</tr>
</tbody>
</table>

3. An Alternate Decomposition of the Waves

The Biot equations have really too many unknown functions when we use both the fluid and the solid displacements. It is better to use just the solid displacement and the fluid pressure, which was pointed out by Bonnets; see also Refs. 23, and 25. To see how this is done recall that the constitutive equation for a Biot-type material are given by

$$\sigma := \lambda \nabla \cdot u I + \mu (\nabla u + (\nabla u)^T) + Q \nabla \cdot v I + s I,$$

where

$$s := Q \nabla \cdot u + R \nabla \cdot v.$$

Here $\sigma$ indicates a second-order tensor and $I$ represents the unit tensor. We may use (9) to eliminate the fluid displacement $v$ from (8). To this end we derive an equation that is satisfied by $s$ by first obtaining from Eq. (9)

$$\nabla s + \omega^2 (\tilde{\rho}_{12} u + \tilde{\rho}_{22} v) = 0$$

and then

$$\triangle s + \frac{\omega^2 \tilde{\rho}_{22}}{R} s + \omega^2 \tilde{\rho}_{22} \left( \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} - \frac{Q}{R} \right) \nabla \cdot u = 0.$$  

Following we identify $s$ as the homogenized pressure and connect it to the pore-pressure, $p_f$ as

$$p_f := -\frac{s}{\beta},$$

where $\beta$ is the porosity.
If \( \sigma_s := \sigma - s I \) then (8) may be written as,
\[
\nabla \cdot \sigma_s + \omega^2 (\tilde{\rho}_{11} u + \tilde{\rho}_{12} v) = 0.
\]
We may again eliminate the water displacement vector \( v \) to obtain the form:
\[
\nabla \cdot \sigma_s + \omega^2 \left( \tilde{\rho}_{11} - \frac{\tilde{\rho}_{12}^2}{\tilde{\rho}_{22}} \right) u - \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} \nabla s = 0,
\]
which may be rewritten in the form:
\[
\left( \lambda + \mu - \frac{Q^2}{R} \right) \nabla (\nabla \cdot u) + \mu \triangle u + \left( \frac{Q}{R} - \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} \right) \nabla s + \omega^2 \left( \tilde{\rho}_{11} - \frac{\tilde{\rho}_{12}^2}{\tilde{\rho}_{22}} \right) u = 0. \tag{17}
\]
For simplicity we refer to \( \tilde{\lambda} := \lambda - \frac{Q^2}{R} \). Following Zimmerman and Stern we introduce the new variable
\[
p := -\frac{s}{\sqrt{\rho_{22}\omega}} \tag{18}
\]
and the representation
\[
u = \nabla \phi_f + \nabla \phi_s + \nabla \wedge \tilde{\psi},
\]
\[
p = A_f \triangle \phi_f + A_s \triangle \phi_s. \tag{19}
\]
By substituting these into (16) and (17) to obtain the equations for the scalar and vector potentials. These are
\[
a_f \triangle \phi_f + \omega^2 \tilde{\rho}_f \phi_f + a_s \triangle \phi_s + \omega^2 \tilde{\rho}_s \phi_s, = 0,
\]
\[
A_f \triangle \phi_f + d_f \phi_f + A_s \triangle \phi_s + d_s \phi_s, = 0, \tag{20}
\]
\[
\triangle \tilde{\psi} + \kappa_2^2 \tilde{\psi} = 0.
\]
Here we are using the notation
\[
a_{f,s} := \tilde{\lambda} + 2\mu + \delta A_{f,s}, \quad \tilde{\rho} := \frac{\tilde{\rho}_{11}\tilde{\rho}_{22} - \tilde{\rho}_{12}^2}{\tilde{\rho}_{22}}. \tag{21}
\]
\[
d_{f,s} := A_{f,s} \frac{\omega^2 \tilde{\rho}_{22}}{R} - \delta, \quad \kappa_2^2 := \frac{\tilde{\rho}_f \omega^2}{\mu}.
\]
where
\[
\delta := \sqrt{\tilde{\rho}_{22}\omega} \left( \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} - \frac{Q}{R} \right).
\]
In order to decouple these waves\(^{a}\) we require that
\[
\frac{d_f}{A_f} = \frac{\tilde{\rho}_f \omega^2}{a_f} \quad \text{and} \quad \frac{d_s}{A_s} = \frac{\tilde{\rho}_s \omega^2}{a_s}. \tag{22}
\]
\(^{a}\)Zimmerman and Stern say for reasons of “compatibility”.

1350001-7
This leads to

\[ A_{f,s} := \frac{-D \pm \sqrt{D^2 + 4(\tilde{\lambda} + 2\mu)\delta^2 \gamma \omega^2}}{2\delta \gamma \omega^2}, \]  

(23)

where

\[ \gamma = \frac{\tilde{\rho}_{22}}{R}, \quad D := (\tilde{\lambda} + 2\mu)\gamma \omega^2 - \delta^2 - \tilde{\rho} \omega^2. \]

The fast and slow compression waves then satisfy

\[ \triangle \phi_{f,s} + \kappa_{f,s}^2 \phi_{f,s} = 0, \]  

(24)

where

\[ \kappa_{f,s}^2 := \frac{d_{f,s}}{A_{f,s}} = \frac{\tilde{\rho} \omega^2}{a_{f,s}}. \]  

(25)

From Eq. (24) traveling plane-waves solutions in the \( x \)-direction, which are independent of the \( y \) direction have the form:

\[ \phi_f(x, z) := f(z)e^{ikx}, \quad \phi_s(x, z) := s(z)e^{ikx} \quad \text{and} \quad \psi := (0, \psi, 0), \quad \text{with} \quad \psi(x, z) := g(z)e^{ikx}, \]

where the coefficients \( f, s, g \) satisfy

\[ f'' - k_f^2 f = 0, \quad \text{where} \quad k_f^2 := k^2 - \kappa_f^2, \]

\[ s'' - k_s^2 s = 0, \quad \text{where} \quad k_s^2 := k^2 - \kappa_s^2, \]

\[ g'' - k_t^2 g = 0, \quad \text{where} \quad k_t^2 := k^2 - \kappa_t^2 \]  

(26)

with \( \kappa_f^2, \kappa_s^2 \) and \( \kappa_t^2 \) defined in (21) and (25), respectively.

### 4. Characteristic Equations for a Viscous Fluid Biot System

For a viscous interstitial fluid in the Biot equations we may consider the fluid to obey a pressure release condition on the surface of the plate. We seek solutions in the wave guide \( \{(x, y, z) : (x, y) \in \mathbb{R}^2, -h \leq z \leq h\} \) in the form of a traveling wave in the \( x \)-direction. We assume “free-surface” boundary conditions on the slab surface \( z = \pm h \); i.e.

\[ \sigma_{zz}(x, \pm h) = 0, \quad \sigma_{xz}(x, \pm h) = 0 \quad \text{and} \quad p(x, \pm h) = 0. \]  

(27)

The resulting solutions are known as Lamb waves. For these waves we assume the vector potential may be written in the form:

\[ \psi = (0, \psi, 0). \]  

(28)

In order to obtain the characteristic equations we need to rewrite \( \sigma_{zz}, \sigma_{xz} \) and \( s \) in terms of the potentials. Using

\[ \epsilon = -\frac{\sqrt{P_{22}} \omega}{R} p - \frac{Q}{R} e = -(A_f \triangle \phi_f + A_s \triangle \phi_s) \frac{\sqrt{P_{22}} \omega}{R} - \frac{Q}{R} \triangle (\phi_f + \phi_s) \]
Lamb Waves in a Poroelastic Plate

we compute

\[
\sigma_{xx} = 2\mu((\phi_f + \phi_s)_{,xx} - \psi_{,xx}) + \left(\lambda - Q^2\frac{A_f\sqrt{p_{22}ω}}{R} - \frac{Q^2}{R}\right) \triangle \phi_f \\
+ \left(\lambda - Q^2\frac{A_s\sqrt{p_{22}ω}}{R} - \frac{Q^2}{R}\right) \triangle \phi_s
\]  

(29)

and

\[
\sigma_{zz} = \lambda(\triangle \phi_f + \triangle \phi_s) + 2\mu(\phi_{f,zz} + \phi_{s,zz} + \psi_{,zz}) + Q\epsilon + s,
\]

but as the pressure vanishes on \(z = \pm h\) it follows that \(Q\epsilon = -\frac{Q^2}{R} \nabla \cdot \mathbf{u}\); hence, the first free-surface boundary condition takes the form

\[
\sigma_{zz}(x,\pm h) = \left(\lambda - \frac{Q^2}{R}\right) (\triangle \phi_f + \triangle \phi_s) + 2\mu(\phi_{f,zz} + \phi_{s,zz} + \psi_{,zz}) = 0.
\]

(30)

Moreover, we obtain the second free-surface boundary condition

\[
\sigma_{xx} = \mu(2\phi_{f,xx} + 2\phi_{s,xx} - \psi_{,xx} + \psi_{,xx}).
\]

(31)

The ordinary differential equations (26) and the free-surface boundary conditions suggest that

\[
f = a_{11} \sinh(k_f z) + a_{12} \cosh(k_f z),
\]

\[
s = a_{21} \sinh(k_s z) + a_{22} \cosh(k_s z),
\]

\[
g = a_{31} \sinh(k_t z) + a_{32} \cosh(k_t z).
\]

Substituting these separated solutions into the boundary conditions (30), (31) leads to the system of differential equations on \(z = \pm h\), namely

\[
\left[\lambda - \frac{Q^2}{R}\right] (f'' - k^2 f) + \tilde{\lambda}(s'' - k^2 s) + 2\mu(k_f^2 f + k_s^2 s + ikg') = 0 \quad \text{at } z = \pm h,
\]

\[
\mu(2ikf' + 2iks' - g'' - k^2 g) = 0 \quad \text{at } z = \pm h.
\]

(33)

To this set we also add the pressure release boundary condition

\[
p = A_f \triangle \phi_f + A_s \triangle \phi_s = 0 \quad \text{at } z = \pm h.
\]

(34)

The first of the Eqs. (33) simplifies to

\[
(-\tilde{\lambda}_f + 2\mu k_f^2) f + (-\tilde{\lambda}_s + 2\mu k_s^2) s + 2i\mu kg' = 0, \quad z = \pm h,
\]

(35)

where \(\tilde{\lambda}_{f,s} = \frac{(\lambda - \frac{Q^2}{R})\rho\omega^2}{a_{f,s}^2}\). Now replacing \(f, s, g\) with their representations (32) we obtain for the second equation

\[
2ik[k_f(a_{11} \cosh(k_f h) \pm a_{12} \sinh(k_f h)) + k_s(a_{21} \cosh(k_s h) \pm a_{22} \sinh(k_s h))] \\
- (k_t^2 + k_s^2)(\pm(a_{31} \sinh(k_t h) + a_{32} \cosh(k_t h))) = 0
\]

(36)
and for the first equation

\[ -\tilde{\lambda}_f [\pm a_{11} \sinh(k_f h) + a_{12} \cosh(k_{f,m} h)] - \tilde{\lambda}_s [\pm a_{21} \sinh(k_s h) + a_{22} \cosh(k_s h)] \\
+ 2\mu (k_f^2 [\pm a_{11} \sinh(k_f h) + a_{12} \cosh(k_f h)] + k_s^2 [\pm a_{21} \sinh(k_s h) + a_{22} \cosh(k_s h)]) \\
+ i k k_l [a_{31} \cosh(k_l) \pm a_{32} \sinh(k_l h)] = 0. \tag{37} \]

From the pressure release condition on the surface we obtain

\[ \frac{A_f}{a_f} (\pm a_{11} \sinh(k_f h) + a_{12} \cosh(k_f h)) + \frac{A_s}{a_s} (\pm a_{21} \sinh(k_s h) + a_{22} \cosh(k_s h)) = 0. \tag{38} \]

Adding and subtracting these equations we may solve for \( a_{11} \) and \( a_{12} \) as

\[ a_{21} = \frac{A_f a_s \sinh(k_f h)}{A_s a_f \sinh(k_s h)} a_{11}, \tag{39} \]
\[ a_{22} = \frac{A_f a_s \cosh(k_f h)}{A_s a_f \cosh(k_s h)} a_{12}. \tag{40} \]

Now substituting these into the other equations allows us to compute

\[ a_{31} = -\frac{\cosh(k_f h) a_{12}}{2i \mu k_{t,m} \cosh(k_l h)} \left[ (-\tilde{\lambda}_f + 2\mu k_f^2) - (-\tilde{\lambda}_s + 2\mu k_s^2) \frac{A_f a_s}{A_s a_f} \right], \tag{41} \]
\[ a_{32} = -\frac{\sinh(k_f h) a_{11}}{2i \mu k_{l} \sinh(k_l h)} \left[ (-\tilde{\lambda}_f + 2\mu k_f^2) - (-\tilde{\lambda}_s + 2\mu k_s^2) \frac{A_f a_s}{A_s a_f} \right]. \tag{42} \]

By setting the determinant of the coefficients of the six equations (36), (37), (38) equal to zero we get the characteristic equations, one giving rise to symmetric modes and the other to antisymmetric modes, namely

\[ [-A_f a_s (-\tilde{\lambda}_s + 2\mu k_{s,m}^2) + A_s a_f (-\tilde{\lambda}_f + 2\mu k_{f,m}^2)] \coth(k_{t,m} h)(k_{t,m}^2 + k_{m}^2) - 4k_{t,m}^2 k_{t,m} \mu \\
\times [A_s a_f k_{f,m} \coth(k_{f,m} h) - A_f a_s k_{s,m} \coth(k_{s,m} h)] = 0, \quad m = 0, 1, 2, \ldots \tag{43} \]

and

\[ [A_f a_s (-\tilde{\lambda}_s + 2\mu k_{s,m}^2) - A_s a_f (-\tilde{\lambda}_f + 2\mu k_{f,m}^2)] \tanh(k_{t,m} h)(k_{t,m}^2 + k_{m}^2) - 4k_{m}^2 k_{t,m} \mu \\
\times [-A_s a_f k_{f,m} \tanh(k_{f,m} h) + A_f a_s k_{s,m} \tanh(k_{s,m} h)] = 0, \quad m = 0, 1, 2, \ldots \tag{44} \]

Equations (43) and (44) can be written in alternative forms as,

\[ L_{f,s} \sinh(k_{t,m} h) \cosh(k_{s,m} h) \cosh(k_{f,m} h) - 4\mu k_{t,m}^2 \cosh(k_{t,m} h) \\
\times [k_{f,m} A_s (k_{s,m}^2 - k_{m}^2) \cosh(k_{s,m} h) \sinh(k_{f,m} h) \\
- k_{s,m} A_f (k_{f,m}^2 - k_{m}^2) \sinh(k_{s,m} h) \cosh(k_{f,m} h)] = 0, \tag{45} \]
where

\[ k \] in the antisymmetric mode eigenvalues. The symmetric and antisymmetric modes for displacement and antisymmetric modes as

\[ \text{Lamb Waves in a Poroleastic Plate} \]

\[
L_{f,s} \cosh(k_{t,m} h) \sinh(k_{s,m} h) \sinh(k_{f,m} h) - 4\mu k_m^2 k_{t,m} \sinh(k_{t,m} h) \\
\times (k_{f,m} A_s (k_{s,m}^2 - k_m^2) \sinh(k_{s,m} h) \cosh(k_{f,m} h) \\
- k_{s,m} A_f (k_{f,m}^2 - k_m^2) \cosh(k_{s,m} h) \sinh(k_{f,m} h)) = 0,
\]

(46)

where

\[
L_{f,s} = [A_s (k_{s,m}^2 - k_m^2) (\tilde{\lambda}(k_{f,m}^2 - k_m^2) + 2\mu k_m^2) - A_f (k_{f,m}^2 - k_m^2) (\tilde{\lambda}(k_{s,m}^2 - k_m^2) + 2\mu k_m^2)] \\
\times (k_{t,m}^2 + k_m^2).
\]

(47)

We notice that these reduce to the elastic case when the porosity vanishes[26,27]

\[
\text{tanh}(\eta_2 h) = \frac{4k^2\eta_1\eta_2}{(2k^2 - \kappa^2)^2}, \quad \text{coth}(\eta_2 h) = \frac{4k^2\eta_1\eta_2}{(2k^2 - \kappa^2)^2},
\]

(48)

where \( \eta_1 = \sqrt{k_i^2 - k_{ep}^2} \) and \( \eta_2 = \sqrt{k_i^2 - k_s^2} \), \( i = 1, \ldots, \infty; \ k_{ep} = \frac{\omega}{c_p}, \ k_s = \frac{\omega}{c_s} \) with \( c_p \) and \( c_s \) being the speed of the longitudinal wave and the shear wave, respectively.

The displacement field components \( u_x \) and \( u_z \) can be expressed in terms of the symmetric and antisymmetric modes as

\[
u_x(x, z) = u_x^s + u_x^a = \pm \sum_{m=0}^{\infty} S_m \phi_{mx}^s(z) e^{\pm ik_m^s x} \pm \sum_{n=0}^{\infty} A_n \phi_{nz}^a(z) e^{\pm ik_n^a x},
\]

\[
u_z(x, z) = u_z^s + u_z^a = \pm \sum_{m=0}^{\infty} S_m \phi_{mx}^s(z) e^{\pm ik_m^s x} \pm \sum_{n=0}^{\infty} A_n \phi_{nz}^a(z) e^{\pm ik_n^a x},
\]

where \( k_m^s \) is the solution of (45) and \( k_n^a \) of (46). Here \( k_m^s \) and \( k_n^a \) refer to the symmetric and antisymmetric mode eigenvalues. The symmetric and antisymmetric modes for displacement in the x-direction are given respectively as

\[
\phi_{mx}^s(z) = ik_m^s \cosh(k_{f,m} z) - ik_m^s \frac{A_f a_s \cosh(k_{f,m} h)}{A_s a_f \cosh(k_{s,m} h)} \cosh(k_{t,m} z) \\
+ \left[ (-\tilde{\lambda}_f + 2\mu k_{f,m}^2) - (-\tilde{\lambda}_s + 2\mu k_{s,m}^2) \frac{A_f a_s}{A_s a_f} \right] \\
\times \frac{\cosh(k_{f,m} h)}{2i\mu k_m^s \cosh(k_{t,m} h)} \cosh(k_{t,m} z)
\]

(49)

and

\[
\phi_{mx}^a(z) = ik_m^a \sinh(k_{f,m} z) - ik_m^a \frac{A_f a_s \sinh(k_{f,m} h)}{A_s a_f \sinh(k_{s,m} h)} \sinh(k_{t,m} z) \\
+ \left[ (-\tilde{\lambda}_f + 2\mu k_{f,m}^2) - (-\tilde{\lambda}_s + 2\mu k_{s,m}^2) \frac{A_f a_s}{A_s a_f} \right] \\
\times \frac{\sinh(k_{f,m} h)}{2i\mu k_m^a \sinh(k_{t,m} h)} \sinh(k_{t,m} z).
\]

(50)
Similarly, the symmetric and antisymmetric modes for displacement in the \( z \)-direction are given respectively as

\[
\varphi_s(z) = k_{f,m} \sinh(k_{f,m} z) - k_{s,m} \frac{A_{fa} \sinh(k_{f,m} h)}{A_{af} \cosh(k_{s,m} h)} \sinh(k_{s,m} z)
\]

\[
- \left[ (-\lambda_f + 2\mu k_{f,m}^2) - (-\lambda_s + 2\mu k_{s,m}^2) \frac{A_{fa}}{A_{af}} \right] \cosh(k_{f,m} h) \sinh(k_{s,m} h) \cosh(k_{t,m} h) \sinh(k_{t,m} z),
\]

\[ \times \frac{\cosh(k_{f,m} h)}{k_{t,m} \cosh(k_{t,m} h)} \sinh(k_{f,m} z). \]  

(51)

\[
\varphi_a(z) = k_{f,m} \cosh(k_{f,m} z) - k_{s,m} \frac{A_{fa} \sinh(k_{f,m} h)}{A_{af} \sinh(k_{s,m} h)} \cosh(k_{s,m} z)
\]

\[
- \left[ (-\lambda_f + 2\mu k_{f,m}^2) - (-\lambda_s + 2\mu k_{s,m}^2) \frac{A_{fa}}{A_{af}} \right] \sinh(k_{f,m} h) \cosh(k_{s,m} h) \sinh(k_{t,m} h) \cosh(k_{t,m} z)
\]

\[ \times \frac{\sinh(k_{f,m} h)}{k_{t,m} \sinh(k_{t,m} h)} \cosh(k_{f,m} z). \]  

(52)

In Eqs. (49) and (51), we have used the notation

\[ k_{q,m}^2 = (k_{m}^s)^2 - \kappa_{q}^2, \quad q = f, s, t \]

whereas in (50) and (52), we have

\[ k_{q,m}^2 = (k_{m}^a)^2 - \kappa_{q}^2, \quad q = f, s, t. \]

5. Orthogonality Relations

In order to establish orthogonality relations for the free boundary poroelastic slab \( \Omega \) we rewrite these equations as a system using the notation of Kupradze.\(^8\) The system has the matrix form:

\[
\mathbf{A}(\partial_x) = \| A_{ij}(\partial_x) \|_{3 \times 3},
\]

where

\[
A_{ij}(\partial_x) = \delta_{ij} \mu \Delta + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j}.
\]

(53)

and

\[
\mathbf{T}(\partial_x, n(x)) = \| T_{ij}(\partial_x, n(x)) \|_{3 \times 3},
\]

where

\[
T_{ij}(\partial_x) = \lambda n_i(x) \frac{\partial}{\partial x_j} + \mu n_j(x) \frac{\partial}{\partial x_i} + \mu \delta_{ij} \frac{\partial}{\partial n(x)}.
\]

(54)

We introduce the four-vector \( \mathbf{U} \):

\[ \mathbf{U} = (u, p) \]
and the matrix differential operator $B(\partial_x, \omega)$ as

$$B(\partial_x, \omega) = \begin{pmatrix} \lambda + 2\mu & \nabla \cdot u - \mu \nabla \times (\nabla \times u) + \alpha \nabla p + \omega^2 \rho u \\ \lambda p + \rho \omega^2 p - \alpha \nabla \cdot u \end{pmatrix}.$$ 

Now let $U_1 = (u_1, p_1)$ and $U_2 = (u_2, p_2)$ be regular vectors in $\bar{\Omega}$ (the closure of $\Omega$) and $Bu_1$ and $BU_2$ be absolutely integrable in $\bar{\Omega}$.

Then we have the identity

$$U_1^* B(\partial_x, \omega) U_2 - U_2^* B(\partial_x, \omega) U_1 = (U_1)^* (Au_2 + \alpha \nabla \cdot p_2 + \rho \omega^2 u_2) + p_1^*(\triangle p_2 + \omega^2 p_2 - \alpha \nabla u_2)$$

$$- (U_2)^* (Au_1 + \alpha \nabla \cdot p_1 + \rho \omega^2 u_1) - p_2^*(\triangle p_1 + \omega^2 p_1 - \alpha \nabla u_1)$$

$$= ((u_1)^* Au_2 - (u_2)^* Au_1) + (p_1^* \triangle p_2 - p_2^* \triangle p_1)$$

$$- \alpha (p_1^* \nabla \cdot u_2 + (u_2)^* \nabla p_1) - \alpha ((u_1)^* \nabla \cdot p_2 + p_2^* \nabla u_1).$$

Since

$$p_1^* \nabla \cdot u_2 = p_1^* \sum_k \frac{\partial u_{2,k}}{\partial x_k} = \sum_k \frac{\partial}{\partial x_k} (p_1^* u_{2,k}) - \sum_k u_{2,k}^* \frac{\partial p_1}{\partial x_k}$$

$$= \sum_k \frac{\partial}{\partial x_k} (p_1^* u_{2,k}) - (u_2)^* \nabla p_1$$

and

$$p_2^* \nabla \cdot u_1 = \sum_k \frac{\partial}{\partial x_k} (p_2^* u_{1,k}) - (u_1)^* \nabla p_2,$$

it follows that

$$U_1^* B(\partial_x, \omega) U_2 - U_2^* B(\partial_x, \omega) U_1 = ((u_1)^* Au_2 - (u_2)^* Au_1) + (p_1^* \triangle p_2 - p_2^* \triangle p_1)$$

$$- \alpha \nabla \cdot (p_1^* \vec{u}_2) + \alpha \nabla \cdot (p_2^* \vec{u}_1).$$

Letting $u_k, k = 1, 2$ be solutions of the differential system and integrating over $D$ and then using the divergence theorem, yields the natural boundary conditions on the surface of the slab, namely

$$\int_D (U_1^* B \vec{u}_2 - U_2^* B \vec{u}_1) dV = \int_S \left\{ [(u_1)^* \vec{T} u_2 - (u_2)^* \vec{T} u_1] \cdot \vec{n} + \left( p_1^* \frac{\partial p_2}{\partial n} - p_2^* \frac{\partial p_1}{\partial n} \right) \\
- \alpha \sum_k p_1^* u_{2,k} n_k - \alpha \sum_k p_2^* u_{1,k} n_k \right\} dS$$

$$= \int_S \left\{ |u_1|^* (\vec{T} u_2 - \alpha p_2) \cdot \vec{n} - |u_2|^* (\vec{T} u_1 - \alpha p_1) \cdot \vec{n} \\
+ \left( p_1^* \frac{\partial p_2}{\partial n} - p_2^* \frac{\partial p_1}{\partial n} \right) \right\} dS = 0,$$
which may be rewritten as
\[ \int_S \left\{ \bar{U}_1^s T \bar{U}_2 + \bar{U}_2^s T \bar{U}_1 \right\} = 0 \] (55)
and
\[ T \bar{U} := T \bar{u} + \nabla p. \] (56)

6. Diffraction of Waves by a Finite Crack

As before we assume an isotropic, poroelastic slab of thickness 2h. The slit (crack) is situated in the plane of symmetry of the plate \( z = 0, \ell_2 \leq x \leq \ell_1 \) of length \( L = |\ell_1 - \ell_2| \). Following Rokhlin, the crack is of infinitesimal thickness yet the opposite sides do not interact. That is there is no transfer of stresses and displacements through the crack. Let the incident Lamb wave arriving from \( x = +\infty \) be described by displacement potentials \( \phi^{\text{inc}} \) and \( \psi^{\text{inc}} \), whereas, the diffracted field will have potentials designated by \( \phi \) and \( \psi \).

The stresses and displacements can be expressed in terms of the potentials
\[ \sigma_{zz} = \lambda [\partial_{xx} \phi_f + \partial_{zz} \phi_f + \partial_{xx} \phi_s + \partial_{zz} \phi_s] + 2\mu [\partial_{zz} \phi_f + \partial_{zz} \phi_s + \partial_{xx} \psi] \]
\[ + \frac{Q_p}{R} - \frac{Q^2}{R} [\partial_x u_x + \partial_z u_z] + s, \] (57)
\[ \sigma_{zz} = \mu [2(\partial_{zz} \phi_f + \partial_{zz} \phi_s) - \partial_{zz} \psi + \partial_{xx} \psi]. \] (58)

Since
\[ \partial_x u_x = \partial_{xx} (\phi_f + \phi_s) - \partial_{zz} \psi, \quad \partial_z u_z = \partial_{zz} (\phi_f + \phi_s) + \partial_{xx} \psi \] (59)
we have, as the pressure vanishes on \( z = \pm h \)
\[ \sigma_{zz} = \left( \lambda - \frac{Q^2}{R} \right) \Delta \phi_f + \left( \lambda - \frac{Q^2}{R} \right) \Delta \phi_s + 2\mu [\partial_{zz} \phi_f + \partial_{zz} \phi_s + \partial_{xx} \psi]. \] (60)

The solution is to be sought in the form:
\[ \phi_f(x, z) = \frac{1}{\sqrt{2\pi}} \int_{i\alpha - \infty}^{i\alpha + \infty} [A_{1,2} \cosh(k_f z) + B_{1,2} \sinh(k_f z)] e^{ikx} \, dk, \]
\[ \phi_s(x, z) = \frac{1}{\sqrt{2\pi}} \int_{i\alpha - \infty}^{i\alpha + \infty} [C_{1,2} \cosh(k_s z) + D_{1,2} \sinh(k_s z)] e^{ikx} \, dk, \]
\[ \psi(x, z) = \frac{1}{\sqrt{2\pi}} \int_{i\alpha - \infty}^{i\alpha + \infty} [E_{1,2} \cosh(k_t z) + F_{1,2} \sinh(k_t z)] e^{ikx} \, dk, \] (61)

where \(-\text{Im}K < \alpha < \text{Im}K\), \(\text{Im}K = \inf\{|Im k_{\nu,m}|\}, \nu = f, s, t\) and
\[ k_{f,m} = \sqrt{k_m^2 - \frac{\omega^2 \rho_0}{\alpha_f}}, \quad k_{s,m} = \sqrt{k_m^2 - \frac{\omega^2 \rho_0}{\alpha_s}}, \quad k_{t,m} = \sqrt{k_m^2 - \frac{\omega^2 \rho_0}{\mu}}. \]
Here the subscript 1 is taken for the region $0 < z \leq h$; whereas, 2 is taken for the
region $-h \leq z < 0$. Six algebraic equations are obtained for obtaining the coefficients
$A_{1,2}(k) \cdots F_{1,2}(k)$ by substituting these into the boundary conditions. For the case of a
viscous interstitial fluid the boundary conditions are
\[
\sigma_{xz}|_{z=\pm h} = 0, \quad \sigma_{zz}|_{z=\pm h} = 0, \quad p|_{z=\pm h} = 0,
\]
assuming the incident wave satisfies the same boundary conditions. In terms of potentials,
these conditions lead to the following equations
\[
2\mu [ik f_{1,2} \sinh(k f z) + ik k_{s} D_{1,2} \cosh(k f z) + ik k_{s} C_{1,2} \sinh(k s z)]
+ i k k_{s} D_{1,2} \cosh(k s z)]|_{z=\pm h} - \mu [E_{1,2}(k f_{1}^{2} + k s^{2}) \cosh(k f z) +
F_{1,2}(k f_{1}^{2} + k s^{2}) \sinh(k s z)]|_{z=\pm h} = 0, \tag{62}
\]
\[
\tilde{\lambda}(k f_{1}^{2} - k s^{2}) [A_{1,2} \cosh(k f z) + B_{1,2} \sinh(k f z)]|_{z=\pm h} + \tilde{\lambda}(k f_{1}^{2} - k s^{2}) [C_{1,2} \cosh(k s z)]
+ D_{1,2} \sinh(k s z)]|_{z=\pm h} + 2 \mu [k_{f_{1}} A_{1,2} \cosh(k f z) + k_{f_{1}} B_{1,2} \sinh(k f z) + k_{s}^{2} C_{1,2} \cosh(k s z)]
+k_{s}^{2} D_{1,2} \sinh(k s z) + ik_{1} k E_{1,2} \sinh(k s z) + ik_{1} k F_{1,2} \cosh(k s z)]|_{z=\pm h} = 0, \tag{63}
\]
where $\tilde{\lambda} = \lambda - \frac{Q^{2}}{2\ell_{1}}$; moreover,
\[
A_{f}(k f_{1}^{2} - k s^{2}) [A_{1,2} \cosh(k f z) + B_{1,2} \sinh(k f z)]|_{z=\pm h}
+ A_{s}(k s^{2} - k s^{2}) [C_{1,2} \cosh(k s z) + D_{1,2} \sinh(k s z)]|_{z=\pm h} = 0. \tag{64}
\]
This will yield six conditions for determining the coefficients. On the middle line we want
continuity of the traction, the normal stress and the pressure:
\[
\sigma_{j z}(x, 0^{+}) - \sigma_{j z}(x, 0^{-}) = 0, \quad j = x, z, \tag{65}
\]
\[
p(x, 0^{+}) - p(x, 0^{-}) = 0. \tag{66}
\]
This yields
\[
2 i k k_{s} D' + 2 i k k_{s} D' - (k f_{1}^{2} + k s^{2}) E' = 0, \tag{67}
\]
\[
A' [\tilde{\lambda}(k f_{1}^{2} - k s^{2}) + 2 \mu k s^{2}] + C' [\tilde{\lambda}(k f_{1}^{2} - k s^{2}) + 2 \mu k s^{2}] + 2 i k k_{s}^{2} E' = 0. \tag{68}
\]
\[
A_{f}(k f_{1}^{2} - k s^{2}) A' + A_{s}(k s^{2} - k s^{2}) C' = 0, \tag{69}
\]
where $A' = A_{1} - A_{2}, B' = B_{1} - B_{2}$ etc.

To complete the system of equations we need three more conditions. These are obtained
in terms of the transforms of the solid and fluid displacements over the slit $[\ell_{2}, \ell_{1}]$. As the
asymptotic behavior of these stresses near the crack tip are known to be integrable\(^{30}\) we
may define the solid displacement in terms of
\[
\tilde{G}_{j}(k, 0) = \frac{1}{\sqrt{2\pi}} \int_{\ell_{2}}^{\ell_{1}} [u_{j}(x, 0^{+}) - u_{j}(x, 0^{-})] e^{-ik x} \, dx, \quad j = x, z. \tag{70}
\]
As the displacement of the solid components are given by 
\[ u_x = \partial_x (\phi_f + \phi_s) + \partial_x \psi, \quad u_z = \partial_z (\phi_f + \phi_s) + \partial_z \psi, \]
we have
\[ \tilde{G}_x(k, 0) = ik(A' + C') - F'k_t \] (71)
and
\[ \tilde{G}_z(k, 0) = k_f B' + k_s D' + E'ik. \] (72)
Likewise, the transform of the jump over the slit of the fluid displacements are
\[ \tilde{H}_j(k, 0) = \frac{1}{\sqrt{2\pi}} \int_{\ell_2} [v_j(x, 0^+) - v_j(x, 0^-)]e^{-ikx} dx, \quad j = x, z; \]
As implied by (15) and (18), the fluid displacement is given by
\[ \mathbf{v} = \frac{\nabla p}{\omega \sqrt{\rho_{22}}} - \frac{\tilde{\rho}_{12}}{\rho_{22}} \mathbf{u} \]
thus we obtain the two equations
\[ \tilde{H}_x(k, 0) = -\frac{\tilde{\rho}_{12}}{\rho_{22}} \tilde{G}_x(k, 0), \] (73)
where we have utilized the pressure continuity condition (69), and
\[ \tilde{H}_z(k, 0) = -\frac{A_f \sqrt{\rho_{22}} \omega}{R} + \frac{Q}{k_f B'} - A_s \sqrt{\rho_{22}} \omega + \frac{Q}{k_s D'} - \frac{\tilde{\rho}_{12}}{\rho_{22}} ikE'. \] (74)
By adding and subtracting the Eqs. (62)–(64) these may be brought into more useful forms:
\[ 2ik[k_f(A'' \sinh(k_f h) + B' \cosh(k_f h)) + k_s(C'' \sinh(k_s h) + D' \cosh(k_s h))] \]
\[ - [E''(k_t^2 + k_s^2) \cosh(k_t h) + (k_t^2 + k_s^2)F'' \sinh(k_t h)] = 0, \] (75)
\[ 2ik[k_f(A' \sinh(k_f h) + B'' \cosh(k_f h)) + k_s(C' \sinh(k_s h) + D'' \cosh(k_s h))] \]
\[ - [E''(k_t^2 + k_s^2) \cosh(k_t h) + (k_t^2 + k_s^2)F'' \sinh(k_t h)] = 0, \] (76)
\[ \tilde{\lambda}_f[(k_f^2 - k_s^2)A' \cosh(k_f h) + (k_f^2 - k_s^2)B'' \sinh(k_f h)] + \tilde{\lambda}_s[(k_s^2 - k_f^2)C' \cosh(k_s h) \]
\[ + (k_s^2 - k_f^2)D'' \sinh(k_s h)] + 2\mu[A'k_f^2 \cosh(k_f h) + B'k_f^2 \sinh(k_f h) + C'k_s \cosh(k_s h) \]
\[ + D''k_s^2 \sinh(k_s h) + ikk_t E'' \sinh(k_t h) + ikk_t F'' \cosh(k_t h)] = 0, \] (77)
\[ \tilde{\lambda}_f[(k_f^2 - k_s^2)A'' \cosh(k_f h) + (k_f^2 - k_s^2)B' \sinh(k_f h)] + \tilde{\lambda}_s[(k_s^2 - k_f^2)C' \cosh(k_s h) \]
\[ + (k_s^2 - k_f^2)D' \sinh(k_s h)] + 2\mu[A'k_f^2 \cosh(k_f h) + B'k_f^2 \sinh(k_f h) + C'k_s \cosh(k_s h) \]
\[ + D'k_s^2 \sinh(k_s h) + ikk_t E' \sinh(k_t h) + ikk_t F' \cosh(k_t h)] = 0, \] (78)
\[ A_f(k_f^2 - k_s^2)[A' \cosh(k_f h) + B' \sinh(k_f h)] \]
\[ + A_s(k_s^2 - k_f^2)[C' \cosh(k_s h) + D' \sinh(k_s h)] = 0, \] (79)
Lamb Waves in a Poroelastic Plate

\[ A_f[(k_f^2 - k^2) A'' \cosh(k_f h) + B' \sinh(k_f h)] + A_s(k_s^2 - k^2)[C'' \cosh(k_s h) + D' \sinh(k_s h)] = 0, \] (80)

where \( A'' = A_1 + A_2, B'' = B_1 + B_2, \) etc.

The goal here is to solve for \( A' \sim F' \) and \( A'' \sim F'' \) in terms of \( \tilde{G}_x, \tilde{G}_z \) and \( \tilde{H}_z \) by using these six equations and Equations (67)–(69), (71), (72) and (74).

Solving (68), (69) and (71) simultaneously, we obtain the following equations

\[ A' = -\frac{\tilde{G}_x A_s(k_s^2 - k^2)2\mu ikk_l}{\Delta_1}, \] (81)

where

\[ \Delta_1(k) = k_l[A_f(k_f^2 - k^2)\{\tilde{\lambda}(k_s^2 - k^2) + 2\mu k^2_s\} - A_s(k_s^2 - k^2)\{\tilde{\lambda}(k_f^2 - k^2) + 2\mu k^2_f\}] 
- 2\mu k^2 s_l[A_f(k_f^2 - k^2) - A_s(k_s^2 - k^2)] \]

\[ C' = \frac{\tilde{G}_x A_f(k_f^2 - k^2)2\mu ikk_l}{\Delta_1}, \] (82)

\[ F' = \frac{\tilde{G}_x}{\Delta_1} \frac{L_{f,s}}{k_f^2 + k^2}, \] (83)

where \( L_{f,s} \) is defined in (47).

Furthermore, (67), (72) and (74) imply

\[ B' = \frac{k_s\{\tilde{G}_z F_s + \tilde{H}_z(k_s^2 - k^2)\}}{\Delta_2}, \] (84)

\[ D' = \frac{k_f\{\tilde{G}_z F_f + \tilde{H}_z(k_f^2 - k^2)\}}{\Delta_2}, \] (85)

\[ E' = \frac{\tilde{G}_z(F_s - F_f)}{\Delta_2} \frac{2\mu ikk_s k_f}{k_f^2 + k^2}, \] (86)

where

\[ F_s = \frac{A_s \sqrt{\rho_{22} \omega} + Q}{R} (k_s^2 + k^2) - \frac{\tilde{\rho}_{12} \omega}{\rho_{22}} 2k^2, \] (87)

\[ F_f = \frac{A_f \sqrt{\rho_{22} \omega} + Q}{R} (k_f^2 + k^2) - \frac{\tilde{\rho}_{12} \omega}{\rho_{22}} 2k^2, \] (88)

\[ \Delta_2(k) = \frac{(A_s - A_f) \sqrt{\rho_{22} \omega}}{R} (k_s^2 - k^2)k_s k_f. \]

If we solve (76), (77) and (79) for \( B'', D'' \) and \( E'' \) in terms of \( A', C' \) and \( F' \) and use (81), (82) and (83), then \( B'', D'' \) and \( E'' \) can be expressed in terms of the unknown function \( \tilde{G}_x \).
as follows:

\[
B'' = 4 \frac{\tilde{G}_x A_s (k_s^2 - k^2) 2 \mu i k_l}{\Delta_1 \Delta_a(2h)} \left[ \Delta_a(h) \cosh k_l \frac{1}{2} h \cosh k_s \frac{1}{2} h \sinh k_f \frac{1}{2} h + \Delta_s(h) \sinh k_l \frac{1}{2} h \sinh k_s \frac{1}{2} h \cosh k_f \frac{1}{2} h \right]
\]

\[
D'' = -4 \frac{\tilde{G}_x A_f (k_f^2 - k^2) 2 \mu i k_l}{\Delta_1 \Delta_a(2h)} \left[ \Delta_a(h) \cosh k_l \frac{1}{2} h \sinh k_s \frac{1}{2} h \cosh k_f \frac{1}{2} h + \Delta_s(h) \sinh k_l \frac{1}{2} h \cosh k_s \frac{1}{2} h \sinh k_f \frac{1}{2} h \right]
\]

\[
E'' = -4 \frac{\tilde{G}_x L_{f,s}}{\Delta_1 (k_f^2 + k^2) \Delta_a(2h)} \left[ \Delta_a(h) \sinh k_l \frac{1}{2} h \cosh k_s \frac{1}{2} h \cosh k_f \frac{1}{2} h + \Delta_s(h) \cosh k_l \frac{1}{2} h \sinh k_s \frac{1}{2} h \sinh k_f \frac{1}{2} h \right]
\]

where

\[
\Delta_a(k, b) = L_{f,s} \cosh k_l \frac{1}{2} b \sinh k_s \frac{1}{2} b \sinh k_f \frac{1}{2} b - 4 \mu k^2 k_l \cosh k_l \frac{1}{2} b
\]

\[
\times \left\{ k_f A_s (k_s^2 - k^2) \sinh k_s \frac{1}{2} b \cosh k_f \frac{1}{2} b - k_s A_f (k_f^2 - k^2) \cosh k_f \frac{1}{2} b \right\}.
\]

\[
\Delta_s(k, b) = L_{f,s} \sinh k_l \frac{1}{2} b \cosh k_s \frac{1}{2} b \cosh k_f \frac{1}{2} b - 4 \mu k^2 k_l \cosh k_l \frac{1}{2} b
\]

\[
\times \left\{ k_f A_s (k_s^2 - k^2) \cosh k_s \frac{1}{2} b \sinh k_f \frac{1}{2} b - k_s A_f (k_f^2 - k^2) \sinh k_f \frac{1}{2} b \right\}.
\]

Quantities \( \Delta_{a,s}(b) \) are characteristic functions for Lamb waves in the layer which are the same as (46) and (45) in Sec. 4.

Likewise, if we solve (75), (78) and (80) for \( A'', C'' \) and \( F'' \) in terms of \( B', D' \) and \( E' \) and use (84), (85) and (86), then \( A'', C'' \) and \( F'' \) can be expressed in terms of the unknown function \( \tilde{G}_z \), and \( \tilde{H}_z \) as follows:

\[
A'' = -4 \left[ \Delta_a(h) \sinh k_l \frac{1}{2} h \sinh k_s \frac{1}{2} h \cosh k_f \frac{1}{2} h + \Delta_s(h) \cosh k_l \frac{1}{2} h \cosh k_s \frac{1}{2} h \sinh k_f \frac{1}{2} h \right]
\]

\[
\times \left\{ \tilde{G}_z F_s + \tilde{H}_z (k_f^2 - k^2) \right\} k_s \frac{1}{2} \Delta_2 \Delta_a(2h) + 2 \mu i k l A_s (k_s^2 - k^2) C_0,
\]

(94)
Lamb Waves in a Poroelastic Plate

\[ C'' = 4 \left[ \Delta_a(h) \sinh k_\frac{1}{2}h \cosh k_\frac{1}{2}h \sinh k_\frac{1}{2}h \cosh k_\frac{1}{2}h + \Delta_s(h) \cosh k_\frac{1}{2}h \sinh k_\frac{1}{2}h \cosh k_\frac{1}{2}h \right] \]

\[ \times \frac{\{ \tilde{G}_z F_f + \tilde{H}_z (k^2_f - k^2) \}}{\Delta_2 \Delta_s(2h)} \frac{k_f}{2} - 2\mu i k k_c A_f (k^2_f - k^2) C_0, \]  

(95)

\[ F'' = -4 \left[ \Delta_a(h) \cosh k_\frac{1}{2}h \sinh k_\frac{1}{2}h \cosh k_\frac{1}{2}h \sinh k_\frac{1}{2}h \cosh k_\frac{1}{2}h \right] \]

\[ \times \frac{\tilde{G}_z (F_s - F_f) 2 i k k_c k_f}{\Delta_2 \Delta_s(2h)(k^2_f + k^2)} - \frac{L_{f,s}}{k^2_f + k^2} C_0, \]  

(96)

where

\[ C_0 = \left[ \tilde{G}_z \{ F_s (\cosh k_f h - \cosh k_i h) - F_f (\cosh k_s h - \cosh k_i h) \} \right. \]

\[ + \tilde{H}_z (k^2_f - k^2) (\cosh k_f h - \cosh k_s h) \]  

\[ \frac{2 i k k_c k_f}{\Delta_2 \Delta_s(2h)}. \]

7. General Solution

To find \( \tilde{G}_z, \tilde{H}_z \) and \( \tilde{H}_z \) we formulate the generalized Wiener–Hopf equation. For this purpose we write the transform of stresses and the pressure in the plane \( z = 0, -\infty < x < \infty \) and make use of the continuity equations (65), (66). This yields, as the asymptotic behavior of these stresses near the crack tip are integrable

\[ \tilde{\sigma}_{xz}(k) = \mu \left[ k i k_f B'' + k i k_s D'' - \frac{(k^2_f + k^2)}{2} E'' \right], \]  

(97)

\[ \tilde{\sigma}_{zz}(k) = \frac{A''}{2} \{ \Lambda_0 (k^2_f - k^2) + 2\mu k^2 \} + \frac{C''}{2} \{ \Lambda_s (k^2_s - k^2) + 2\mu k^2 \} + i k k_c \mu F'', \]  

(98)

\[ \tilde{p}(k) = A_f (k^2_f - k^2) \frac{A''}{2} + A_s (k^2_s - k^2) \frac{C''}{2}, \]  

(99)

where

\[ \Lambda_{f,s} = \lambda - \left( \frac{Q}{R} + 1 \right) \sqrt{\rho_{22} \omega A_{f,s}} - \frac{Q^2}{R}. \]

Thus if we put the values of \( B'', D'' \) and \( E'' \) given by (89), (90) and (91) into (97), we obtain following equation

\[ \tilde{\sigma}_{xz} = 4\mu \tilde{G}_z L_x(k), \]  

(100)

where

\[ L_x(k) = \frac{\Delta_a(k, h) \Delta_s(k, h)}{\Delta_1(k) \Delta_a(k, 2h)}. \]  

(101)
The total stresses on the surface of the crack is zero whence it follows that

\[ \sigma_{xx} = -\sigma_{xx}^{inc} \quad \text{and} \quad \sigma_{zz} = -\sigma_{zz}^{inc} \quad \text{at} \quad z = 0 \quad \text{and} \quad \ell_2 \leq x \leq \ell_1. \]  

Suppose the incident wave contains the asymmetric mode such that

\[ \phi_{inc}^{\ell} = A_p \sinh(k_f z) e^{ik_p x} \]  

i.e. \( a_{11} = A_p \neq 0 \) and \( a_{12} = 0 \) in (32). Then (40) and (41) imply \( a_{22} = a_{31} = 0 \). As a result of (39) and (42), the three potentials for the incident asymmetric waves are determined by \( A_p \) and \( k_p \).

Boundary condition (105) yields

\[ \sigma_{xx}^{\ell} = -\frac{W_x}{i(k_p - k)\sqrt{2\pi}} \left[ e^{i(k_p - k)\ell_1} - e^{i(k_p - k)\ell_2} \right], \]  

where \( W_x \) is given by

\[ W_x = A_p 2\mu ik k_f \left\{ \frac{1 - \cosh k_f h}{\cosh k_l h} \right\} \left\{ \frac{1 - \cosh k_s h}{\cosh k_l h} \right\} \left( A_f a_{22} k_f \sinh k_f h \right). \]  

Upon substitution of (102), the functional equation (100) becomes

\[ e^{-ik\ell_2} \tilde{\sigma}_{xx}^+ + \tilde{\sigma}_{xx}^\ell + e^{-ik\ell_1} \tilde{\sigma}_{xx}^- = 4\mu \tilde{G}_x L_x(k). \]  

Equation (109), which is the generalized functional Wiener–Hopf equation, contains three unknown functions: \( \tilde{\sigma}_{xx}^+, \tilde{\sigma}_{xx}^- \) and \( \tilde{G}_x \). Quantities \( \tilde{\sigma}_{xx}^\pm \) are analytic in the upper (lower) half-planes of the complex variable \( k \) and \( \tilde{G}_x \) is a regular function:

\[ \tilde{G}_x = \frac{1}{\sqrt{2\pi}} \int_{\ell_2}^{\ell_1} G_x(x) e^{-ikx} dx. \]  

Introducing the notation

\[ F_x^+(k) = \tilde{\sigma}_{xx}^+ + \frac{W_x e^{ikp\ell_2}}{i\sqrt{2\pi(k_p - k)}}, \]  

\[ F_x^-(k) = \tilde{\sigma}_{xx}^- - \frac{W_x e^{ikp\ell_1}}{i\sqrt{2\pi(k_p - k)}}, \]
we rewrite (109) in the form:
\[ e^{-ikt} F_x^+(k) + e^{-ikt} F_x^-(k) = 4\mu \tilde{G}_x L_x^+(k)L_x^-(k). \]  
(113)

The representation of function \( L_x(k) \) in the form of two multipliers
\[ L_x(k) = L_x^+(k)L_x^-(k), \]  
(114)
is termed factoring. The function \( L_x(k) \) is regular and does not have zeros in the band 
\(-\text{Im}K < \text{Im}k < \text{Im}K\); Function \( L_x^+(k) \) is regular and does not have zeros in the upper 
half-plane \( \text{Im}k > -\text{Im}K \); \( L_x^-(k) \) is regular and does not have zeros in the lower half-plane 
\( \text{Im}k < \text{Im}K \). Here, as usual \( L_x^+(k) \) and \( L_x^-(k) \) are given in terms of \( L_x \)
\[ \ln L_x^+(k) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \ln L_x(\zeta) \frac{d\zeta}{\zeta - k}, \quad (\text{Im}K > c), \]  
(115)
\[ \ln L_x^-(k) = -\frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \ln L_x(\zeta) \frac{d\zeta}{\zeta - k}, \quad (-\text{Im}K < c). \]  
(116)
Note that although \( L_x(\zeta) \sim \sqrt{\zeta} \) as \( \zeta \to \infty \) in the strip, the integrals are convergent if taken
in the sense
\[ \lim_{T \to \infty} \int_{-T+ir}^{T+ir} \{ \} d\zeta. \]  
(117)

Letting \( c \) approach zero in (115) and (116), we finally obtain
\[ \ln L_x^+(k) = \frac{1}{2} \ln L_x(k) + \frac{k}{\pi i} \int_0^\infty \ln L_x(\zeta) \frac{d\zeta}{\zeta^2 - k^2}, \]  
(118)
\[ \ln L_x^-(k) = \frac{1}{2} \ln L_x(k) - \frac{k}{\pi i} \int_0^\infty \ln L_x(\zeta) \frac{d\zeta}{\zeta^2 - k^2}. \]  
(119)

We multiply (113) by \( e^{ikt}/L_x^+(k) \), and perform the expansion
\[ \frac{F_x^-(k)}{L_x^-(k)} e^{-ikt_1 - t_2} = V^+(k) + V^-(k), \]  
(120)
where
\[ V^+(k) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} e^{-i\zeta(t_1 - t_2)} F_x^-(\zeta) \frac{d\zeta}{(\zeta - k)L_x^+(\zeta)}, \]  
(121)
\[ V^-(k) = -\frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} e^{-i\zeta(t_1 - t_2)} F_x^-(\zeta) \frac{d\zeta}{(\zeta - k)L_x^+(\zeta)}. \]  
(122)
for \( \text{Im}K > b > a > -\text{Im}K \). We now reduce (113) to the form:
\[ \left\{ \begin{array}{c}
F_x^+(k) \\
L_x^+(k)
\end{array} \right\} - \left\{ \begin{array}{c}
W_x e^{ik_p t_2} \\
i \sqrt{2\pi(k_p - k)L_x^+(k_p)}
\end{array} \right\} V^+(k) = 4\mu \tilde{G}_x L_x(k)e^{ikt_2} \]
\[ - V^-(k) - \left\{ \begin{array}{c}
W_x e^{ik_p t_2} \\
i \sqrt{2\pi(k_p - k)L_x^+(k_p)}
\end{array} \right\} V^-(k). \]  
(123)
The functional equation obtained above is solved by the Wiener–Hopf procedure. As a result we obtain
\[
\frac{F^+_x(k)}{L^+_x(k)} + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{-i\zeta(t_1-t_2)} F^+_x(\zeta) \frac{d\zeta}{(\zeta-k)L^+_x(\zeta)} - i\sqrt{2\pi}W_x e^{ikp} = 0.
\] (124)

Upon multiplying (113) by \(e^{i\ell_1L_x(k)}\), we obtain
\[
\frac{F^+_x(k)}{L^+_x(k)} - \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{i\zeta(t_1-t_2)} F^+_x(\zeta) d\zeta = 0.
\] (125)

Replacing \(\zeta\) in (124) by \(-\zeta\), replacing \(k\) in (125) by \(-k\), and setting
\[
S^+_1(k) = F^+_x(k) + F^+_x(-k), \quad S^+_2(k) = F^+_x(k) - F^+_x(-k),
\] (126)
we obtain
\[
\frac{S^+_1(k)}{L^+_x(k)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{i\zeta(t_1-t_2)} S^+_2(\zeta) \frac{d\zeta}{(\zeta+k)L_x(\zeta)} - i\sqrt{2\pi}W_x e^{ikp} = 0.
\] (127)

where 0 \(\leq c < |\text{Im}K|\), and one takes either subscript 1 and the upper sign, or subscript 2 and the lower sign.

Introducing the notation
\[
S^*_1 = \frac{X_{1,2}L^+_x(k)L_x(\zeta)}{2\pi i},
\]
and letting \(c \to 0\) we obtain the following Fredholm integral equation,
\[
X_{1,2}(k) = e^{-i\zeta(t_1-t_2)}X_{1,2}(-k)X_{1,2}(-k) + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} K(k,\zeta)X_{1,2}(\zeta)d\zeta - \frac{1}{(k_p-k)} = 0,
\] (129)
where the kernel \(K(k,\zeta)\) is given by
\[
K(k,\zeta) = \frac{e^{i\zeta(t_1-t_2)L_x(\zeta)}}{(\zeta+k)L_x(\zeta)}.
\] (130)

From (98) and (99) we obtain following two equations, whose solutions are obtained in exactly the same way as it was done previously.
\[
\bar{\sigma}_{zz} = -2G_z \frac{\Delta_a(k,h)K(k,h) + \Delta_a(k,h)L(k,h)}{\Delta_2\Delta_a(k,2h)} - 2H_z(k_i^2-k^2) \frac{\Delta_a(k,h)M(k,h) + \Delta_a(k,h)N(k,h)}{\Delta_2\Delta_a(k,2h)},
\] (131)
\[
\bar{p}(k) = -2G_z \frac{\Delta_a(k,h)P(k,h) + \Delta_a(k,h)Q(k,h)}{\Delta_2\Delta_a(k,2h)} - 2H_z(k_i^2-k^2) \frac{\Delta_a(k,h)R(k,h) + \Delta_a(k,h)S(k,h)}{\Delta_2\Delta_a(k,2h)}.
\] (132)
where

\[ K(k, h) = \sinh \frac{k_1 h}{2} \left( F_s k_s (2\mu k_f^2 - \tilde{\Lambda}_f) \cosh \frac{k_f h}{2} \sinh \frac{k_s h}{2} - F_f k_f (2\mu k_f^2 - \tilde{\Lambda}_s) \sinh \frac{k_f h}{2} \cosh \frac{k_s h}{2} \right) \]

\[ - \cosh \frac{k_1 h}{2} \sinh \frac{k_f h}{2} \sinh \frac{k_s h}{2} \frac{4\mu k_s^2 k_1 k_s k_f (F_s - F_f)}{k_f^2 + k_s^2}, \]

\[ L(k, h) = \cosh \frac{k_1 h}{2} \left( F_s k_s (2\mu k_f^2 - \tilde{\Lambda}_f) \sinh \frac{k_f h}{2} \cosh \frac{k_s h}{2} - F_f k_f (2\mu k_f^2 - \tilde{\Lambda}_s) \cosh \frac{k_f h}{2} \sinh \frac{k_s h}{2} \right) \]

\[ - \sinh \frac{k_1 h}{2} \cosh \frac{k_f h}{2} \cosh \frac{k_s h}{2} \frac{4\mu k_s^2 k_1 k_s k_f (F_s - F_f)}{k_f^2 + k_s^2}, \]

\[ M(k, h) = \sinh \frac{k_1 h}{2} \left( k_s (2\mu k_f^2 - \tilde{\Lambda}_f) \cos \frac{k_f h}{2} \sinh \frac{k_s h}{2} - k_f (2\mu k_f^2 s - \tilde{\Lambda}_s) \sinh \frac{k_f h}{2} \cosh \frac{k_s h}{2} \right) \],

\[ N(k, h) = \cosh \frac{k_1 h}{2} \left( k_s (2\mu k_f^2 - \tilde{\Lambda}_f) \sin \frac{k_f h}{2} \cosh \frac{k_s h}{2} - k_f (2\mu k_f^2 s - \tilde{\Lambda}_s) \cosh \frac{k_f h}{2} \sinh \frac{k_s h}{2} \right) \],

\[ P(k, h) = \sinh \frac{k_1 h}{2} \left( F_s k_s A_f (k_f^2 - k^2) \cosh \frac{k_f h}{2} \sinh \frac{k_s h}{2} - F_f k_f A_s (k_s^2 - k^2) \sinh \frac{k_f h}{2} \cosh \frac{k_s h}{2} \right) \],

\[ Q(k, h) = \cosh \frac{k_1 h}{2} \left( F_s k_s A_f (k_f^2 - k^2) \sin \frac{k_f h}{2} \cosh \frac{k_s h}{2} - F_f k_f A_s (k_s^2 - k^2) \cosh \frac{k_f h}{2} \sinh \frac{k_s h}{2} \right) \],

\[ R(k, h) = \sinh \frac{k_1 h}{2} \left( k_s A_f (k_f^2 - k^2) \cosh \frac{k_f h}{2} \sinh \frac{k_s h}{2} - k_f A_s (k_s^2 - k^2) \sinh \frac{k_f h}{2} \cosh \frac{k_s h}{2} \right) \],

\[ S(k, h) = \cosh \frac{k_1 h}{2} \left( k_s A_f (k_f^2 - k^2) \sin \frac{k_f h}{2} \cosh \frac{k_s h}{2} - k_f A_s (k_s^2 - k^2) \cosh \frac{k_f h}{2} \sinh \frac{k_s h}{2} \right) \]

with \( \tilde{\Lambda}_{f,s} := \Lambda_{f,s} k_f^2 k_s^2 \).

8. Concluding Remarks

The Fredholm equation (129) permits us to solve the Toy Problem with a free boundary condition. For a poro-elastic plate this boundary condition is not physical. However, the solution of this homogeneous problem permits us to solve the physical problem with Biot conditions for a porous plate abutting onto a fluid regions above and below. This physical problem will be solved analytically in a subsequent paper including the necessary numerics.

Acknowledgments

Robert P. Gilbert’s This research was supported in part by the National Science Foundation through grant DMS-0920850 and M. Yvonne Ou’s This research was supported in part by the National Science Foundation through grant DMS-0920852 and UD Pilot NIH COBRE grant.
References

10. R. P. Gilbert and A. Mikelić, Homogenizing the acoustic properties of the seabed, part I.
Lamb Waves in a Poroelastic Plate


