Poroelastic Wave Propagation Using CLAWPACK

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Talk Abstract
We use the CLAWPACK (Conservation LAws PACK-age) [1] finite volume method code developed at the University of Washington to solve Biot’s equations [2] [3] for dynamics of a porous, fluid-saturated elastic medium. These equations were developed to model fluid-saturated rock formations, but are also applicable to other porous solids, such as in vivo bone [4]. The wave structure of the solutions is similar to that of anisotropic elasticity, but the presence of the fluid gives rise to a second, much slower P-wave, in addition to the standard fast P-wave and S-wave. The viscous drag between the fluid and the solid matrix appears as a source term, and causes the system to be dispersive. We present numerical results for a selection of test cases from the literature.

Introduction
Maurice A. Biot’s developed his theory of mechanics of a porous, fluid-saturated elastic medium over the period between 1935 and 1957; a summary of much of his work can be found in [2] and [3]. Biot theory uses linear elasticity to describe the solid portion of the medium (often termed the skeleton), the Navier-Stokes equations to describe the fluid portion, and Darcy’s law to model the aggregate motion of the fluid through the skeleton. While it was originally developed to model fluid-saturated rock formations, Biot theory has also been used to describe in vivo bone [4].

In this work, Biot’s equations were numerically solved using the CLAWPACK finite volume code developed at the University of Washington [1]. CLAWPACK implements high-resolution finite volume methods for wave-propagation problems, which are based on solving a Riemann problem at each cell interface, then applying second-order correction terms, with limiters to prevent unphysical oscillations in the solution that would otherwise be present.

Biot’s equations
If we assume a two-dimensional problem in the $xz$-plane involving a homogeneous (on the macroscopic scale), orthotropic medium where the principal material axes coincide with the coordinate axes, after some effort Biot’s equations can be written as a first-order hyperbolic system with a source term,

$$\partial_t q + A \partial_x q + B \partial_z q = Dq$$

(1)

The state vector $q$ and the coefficient matrices $A$, $B$, and $D$ are defined as follows:

$$q = [\sigma_{xx}, \sigma_{zz}, \sigma_{xz}, v_x, v_z, p, q_x, q_z]^T$$

$$A = \begin{bmatrix}
0 & 0 & 0 & c_{11}^x & 0 & 0 & 0 & \alpha_1 M \\
0 & 0 & 0 & c_{11}^x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{55}^z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha_1 M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\kappa_1}{\Delta_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$B = \begin{bmatrix}
0 & 0 & 0 & c_{13}^x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{13}^x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{53}^z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha_3 M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\kappa_3}{\Delta_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$D = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

(2)

The state variables in $q$ are the normal and shear stresses in the $xz$-plane ($\sigma_{xx}, \sigma_{zz}, \sigma_{xz}$), the velocity components of the skeleton relative to an inertial frame ($v_x, v_z$), the fluid pressure ($p$), and the velocity components of the fluid relative to the skeleton ($q_x, q_z$). The parameters in (2) are derived from the properties of the constituent materials: $\rho_f$ is the fluid density; $\eta$ is the fluid viscosity; $\rho$ is the total material density (the volume-weighted average of the solid density $\rho_s$ and the fluid density); $\kappa_1$ and $\kappa_3$ are the permeabilities in the $x$ and $z$ directions; $\Delta_1$ and $\Delta_3$ are derived quantities relating to relative densities.
\[ \Delta_j = \rho m_j - \rho^2_f; \]  
\[ m_1 \text{ and } m_3 \text{ are effective inertias for motion of the fluid in the } x \text{ and } z \text{ directions, } m_j = \rho f T_j/\phi; \]  
\[ T_1 \text{ and } T_3 \text{ are the tortuosities of the medium in the } x \text{ and } z \text{ directions, and give the factor by which the kinetic energy of the fluid must be higher in order to flow with a given aggregate velocity through the convoluted pore structure of the skeleton; } \phi \text{ is the volume fraction of the material occupied by the pores (specifically, the connected pore structure through which the fluid is free to move); } M \]  
\[ = K_s^2/(D - (2c_{11} + c_{33} + 2c_{12} + 4c_{13})/9); \]  
\[ D \text{ is an additional effective compressibility parameter, } D = K_s(1 + \phi(K_s/K_f - 1)); \]  
\[ K_s \text{ and } K_f \text{ are the bulk moduli of the solid and fluid; } c_{ij}^s \text{ are the components of the undrained elastic stiffness tensor, with the needed values given by } c_{ij}^s = c_{ij} + \alpha_i \alpha_j M \text{ for } i, j = 1, 3 \text{ and } c_{55}^s = c_{55}; \]  
\[ \alpha_1 \text{ and } \alpha_3 \text{ are given by } \alpha_1 = 1 - (c_{11} + c_{12} + c_{13})/(3K_s) \text{ and } \alpha_3 = 1 - (2c_{13} + c_{33})/(3K_s); \]  
\[ \text{and } c_{ij} \text{ are the components of the "drained" elastic stiffness tensor of the skeleton (disregarding the presence of the fluid).} \]

Since this system of equations is hyperbolic, its solution can be described in terms of waves. For an inviscid fluid \((\eta = 0 \text{ and thus } D = 0)\), three families of waves arise: a fast P-wave analogous to a standard elastic P-wave, in which the fluid and skeleton move in phase with each other; a shear wave analogous to an elastic S-wave; and a slow P-wave, in which the fluid and skeleton move 180 degrees out of phase with each other. In the presence of viscosity, the same wave structure exists, but the fast P-wave and S-wave become slightly dispersive, and the slow P-wave becomes strongly dispersive. The slow P-wave is also heavily damped in the presence of viscosity.

**Solution process**

System (1) is a combination of the purely hyperbolic system \(\partial_t q + A \partial_x q + B \partial_y q = 0\) and the system of ODEs \(\partial_t q = D q\), so the combined system was solved by using Godunov operator splitting. Although this splitting method is formally only first-order accurate, in practice the coefficient of the first-order error term is typically low enough that it does not harm overall solution accuracy even when used with second-order methods for the individual operators.

Since the source part of the equation (1) is a linear, constant-coefficient system of ODEs, it was solved analytically; this has the advantages both of giving the highest available accuracy to that part of the solution process, and having no timestep size restriction as would be the case with some numerical solution techniques. For the hyperbolic part of (1), the Riemann problems arising at each cell interface were solved to machine precision using eigenvalue decompositions of the \(A\) and \(B\) matrices obtained from LAPACK. The MC limiter was used for all families of waves.

**Test cases**

To validate our work, we compare against the results of de la Puente [5]. All test cases presented here are for a problem with initial condition \(q(x, z, 0) = 0\), with outflow boundary conditions on all sides of the domain, implemented using zero-order extrapolation. Excitation is provided by a point source located at \((x_{src}, z_{src})\), having a Ricker wavelet time profile (delayed by an amount \(t_{src}\) so that it has negligible magnitude at the start of the simulation) with peak frequency \(f_{src}\), and acting with peak intensity \(1 \text{ Pa} \cdot \text{m}^2\) on the fluid pressure \(q_f\) in the state vector) and \(-1 \text{ Pa} \cdot \text{m}^2\) on the \(z\)-direction normal stress \(q_z\). The problem domain for each test case is a rectangle extending from \(x = x_{min} \text{ to } x_{max}\) and from \(z = z_{min} \text{ to } z_{max}\). Material properties and other simulation parameters for the various cases are given in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3 (sandstone)</th>
<th>Case 3 (shale)</th>
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<td>(K_s) (GPa)</td>
<td>80.9</td>
<td>80.9</td>
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<td>(c_{33}) (GPa)</td>
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<td>1.2</td>
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<td>53.4</td>
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<td>(\phi)</td>
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<td>0.2</td>
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<td>(\kappa_1) (10(^{-15}) m(^2))</td>
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<td>100</td>
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<td>100</td>
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<td>100</td>
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<tr>
<td>(T_1)</td>
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<td>2</td>
<td>2</td>
<td>2</td>
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<td>(T_3)</td>
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<td>2.5</td>
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<td>1040</td>
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<td>(\eta) (10(^{-3}) kg/m/s)</td>
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<td>(f_{src}) (Hz)</td>
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<td>9.35</td>
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</table>

The first test case considers an orthotropic sandstone bed saturated with brine, and neglects viscosity; it was run on a uniform \(501 \times 501\) cell Cartesian grid. Figures 1 and
show the skeleton velocity field 1.56 ms after the start of the simulation, and are analogous to figures 7(a) and 7(b) of de la Puente. (The black dot in each figure denotes the source location.) All three waves (fast P-wave, S-wave, and slow P-wave) are clearly visible; the wavefronts are not circular because of the anisotropy of the material.

![Figure 1: x component of skeleton velocity for test case number 1.](image1)

![Figure 2: z component of skeleton velocity for test case number 1.](image2)

The second test case is identical to the first, with the exception that viscosity is included; the skeleton velocity field is shown in figures 3 and 4, analogous to figures 7(c) and 7(d) of de la Puente. In this case, the slow P-wave is essentially eliminated by the viscous drag between the skeleton and the fluid, and the amplitudes of the other waves are somewhat reduced.

The third test case is for a layered bed of shale and sandstone, and demonstrates wave reflection and interconversion between wave families at the material interface. This test was run with using the AMRCLAW package in CLAWPACK, which implements Berger-Colella-Oliger Cartesian-grid adaptive mesh refinement; the coarsest grid was $70 \times 75$ cells, and there were two levels of finer grids, with refinement factors of 4 for the intermediate grids and 6 for the finest grids. The skeleton $x$-direction velocity field 0.25 seconds after the start of the simulation is shown in figure 5, analogous to figure 9(a) of de la Puente. The faint light rectangles show the AMR grid boundaries, and the white-centered black dots show the locations of “gauges” where the time-history of the solution was recorded in detail. Figures 6 and 7 show the time-history of the $x$ and $z$ components of skele-
ton velocity at the uppermost gauge, located at the point (950m, 750m), analogous to figures 10(a) and 10(b) of de la Puente; due to space constraints, no other gauge data are included here.

![Figure 5: $z$ component of skeleton velocity for test case number 3](image)

![Figure 6: History of $x$ component of skeleton velocity at the uppermost gauge in test case number 3](image)

![Figure 7: History of $z$ component of skeleton velocity at the uppermost gauge in test case number 3](image)

References


