NOTES ON RATIONAL AND REAL NUMBERS

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The notion of a number is as old as mathematics itself, and their developments have been inseparable. Usually a new set of numbers included the old set, or, as we often say, extended it. In this way, the set $\{1, 2, 3, ...\}$ of natural numbers, has been extended by adding zero to it, negative integers, rational numbers (fractions), irrational numbers, complex numbers. The actual extensions happened not in the order suggested by the previous sentence, and the history of the process is fascinating. The main motivation for this extension came from mathematics itself: having a greater set of numbers allowed mathematicians to express themselves with better precision and fewer words, i.e., with greater ease. Numbers form one of the most important part of mathematical "language", and in this regard, their development is very similar to the development of live languages, where the vocabulary increases mostly for convenience, rather than of necessity.

It is very hard to part with conveniences after getting used to them. Imagine the world without electricity, or even worse – the mathematics without real numbers. Then objects like $\sqrt{5}$, or $\sin 10°$, or $\log_{10} 7$ would cease to exist, as they do not exist among rational numbers. The use of quadratic equations, or Trigonometry, or Calculus would terminate ... Well, enough of this nightmare.

It is often hard to define basic mathematical notions. The rigor of such definition depends on the time they are made, and of the depth of the subject where they are used. For example, a better definition of a function became important with the development of Calculus, and of abstract algebra. Often the development of mathematical techniques and accumulation of mathematical facts far preceded the thorough discussion of the objects being studied. As examples, one can mention integers, functions, limits, rational and complex numbers. It took many centuries between their appearance and use in mathematics, and the time when they were defined at the level meeting now days standards. Of course, mathematics is not special in this regard. For thousands of years comedies were played in theaters, but definitions of humor, or of the notion of 'funny' appeared very recently. One may wonder whether those were needed at all, but very few mathematicians will doubt that in order to discuss rational and irrational numbers it is important to first define them.

These notes are motivated by the desire to clarify for myself what I wish to say about the rationals and reals in the courses I teach. There are many accounts of these topics in the literature, but I have difficulties of using them in my courses. Either the exposition is too long, or too short, or at the kindergarten level. Or it requires greater mathematical maturity from the students, or it does not mention things I find necessary to be mentioned, or I disagree with the emphases, or with

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the methodology .... These are, probably, typical reasons for one to begin writing on a classical topic. We often believe that our choices and tastes are better, and if not, then at least our writings will help students in *our* courses. Teachers are like actors, and there are as many Pythagorean theorems as there are Hamlets. So, here we are.

1. **Rational numbers.**

Suppose we are familiar with natural numbers

$$\mathbb{N} = \{1, 2, \ldots, n, \ldots\},$$

and with integers

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots, n, -n, \ldots\}.$$

We accept the properties of operations and of order on $\mathbb{Z}$ as known.

All similar properties of rational numbers, which we introduce below, will follow easily from their definition and the corresponding properties of integers.

Construction of rational numbers begins with a set of symbols, called **fractions**, and denoted by

$$\frac{m}{n}, \text{ or } m/n,$$

where $m, n \in \mathbb{Z}$ and $n \neq 0$. Two fractions m/n and m'/n' are defined to be **equal**, if they are precisely the same, i.e., if and only if $m = m'$ and $n = n'$. On the other hand, every one knows that fractions describe certain objects, like parts of an apple, and that $1/2$, $2/4$ or $3/6$ of the same apple are indistinguishable. That is why we write $1/2 = 2/4 = 3/6$ as obvious, which contradicts our definition of equalities of fractions. Why do symbols which look very different are set to be equal? This is the main reason for the discussion below.

1.1. **Let's complicate things.**

The standards of rigor in modern mathematics often require reductions of notions to the set-theoretic language. Usual definition of a set does not allow equal (same as identical) elements. Notions like an ordered pair of objects, a relation on a set, a function, etc., are parts of this language. In what follows we describe how rational numbers can be built from integers in this way.

This and the next two subsections are for the readers who are familiar with the notion of an equivalence relation on a set, and with basic facts related to it. If you are not one of these readers, just browse through the subsections.

We say that a fraction a/b is **equivalent** to a fraction c/d, and write it as $a/b \sim c/d$ if and only if $ad = bc$ and $b, d \neq 0$.[1] Thus the equivalence of new objects (fractions) is defined in terms of equality of familiar objects, namely integers.

Examples.

(i) $10/15 \sim 22/33$, since $10 \cdot 33 = 330 = 15 \cdot 22$.
(ii) $at/bt \sim a/b$ for all $t \in \mathbb{Z}, t \neq 0$
(iii) Every fraction a/b is equivalent to a fraction a'/b', where $\gcd(a', b') = 1$. In order to prove this, write $a = a'r$ and $b = b'r$, where $r = \gcd(a, b)$. E.g., $10/15 \sim 22/33 \sim 2/3$.

It is easy to check that the just defined relation $\sim$ satisfies the following properties: for all fractions

- $a/b \sim a/b$ (reflexive property)
- if $a/b \sim c/d$, then $c/d \sim a/b$ (symmetric property)
- if $a/b \sim c/d$ and $c/d \sim e/f$, then $a/b \sim e/f$ (transitive property).

Let us prove the last property. As $a/b \sim c/d$ and $c/d \sim e/f$, we have $ad = bc$ and $cf = de$. Multiplying both sides of the first equality by f, and of the second by b, we obtain $adf = bcf$ and $bcf = bde$. This implies $adf = bde$, and, as $d \neq 0$, we have $af = be$. As $b, f \neq 0$, we get $a/b \sim e/f$.

The readers who are qualified to read this subsection, recognize that the three properties above mean that $\sim$ is an equivalence relation on the set of fractions. It is a standard fact that whenever we have an equivalence relation on a set, the set becomes partitioned into the equivalence classes. In our case, the equivalence class containing a/b, which we denote by $[a/b]$, is the following set

$$[a/b] = \{c/d : c/d \sim a/b\},$$

and classes $[p/q]$ and $[x/y]$ are equal if and only if $p/q \sim x/y$. For example,

$$[1/2] = [-3/-6] = [1000/2000] =$$

$$\{1/2,\ -1/-2,\ 2/4,\ -2/-4, \ldots, 1000/2000, \ldots\}.$$

Clearly, $[at/bt] = [a/b]$ for every $t \in \mathbb{Z}, t \neq 0$.

Now we define addition and multiplication of the equivalence classes in the following way:

$$\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] = \left[\frac{ad + bc}{bd}\right] \quad \text{and} \quad \left[\frac{a}{b}\right] \cdot \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right].$$

It is not obvious that our definition of addition and multiplication of classes is **well-defined** in the following sense: since we have different notations for the same class, like $[2/4] = [3/6]$ and $[-2/7] = [-10/35]$, how can we be sure that

$$[2/4] + [-2/7] = [3/6] + [-10/35]?$$

[1] If one does not think that symbols like a/b belong to the language of set theory, one is welcome to use (a, b) instead.

In other words, how can we be sure that the result of this newly defined addition doesn't depend on the choices of fractions representing the summands? Nevertheless the last equality is true. Indeed,

$$[2/4] + [-2/7] = \left[\frac{14 + (-8)}{28}\right] = \left[\frac{6}{28}\right] = \left[\frac{3}{14}\right],$$

$$[3/6] + [-10/35] = \left[\frac{105 + (-60)}{210}\right] = \left[\frac{45}{210}\right] = \left[\frac{3}{14}\right].$$

Let us prove now that the addition is well-defined. Let $[a/b] = [a'/b']$ and $[c/d] = [c'/d']$. Then $[a/b] + [c/d] = [(ad + bc)/bd]$ and $[a'/b'] + [c'/d'] = [(a'd' + b'c')/b'd']$. We have to show that $[(ad+bc)/bd] = [(a'd'+b'c')/b'd']$. Indeed, remembering that $b, d, d, d' \neq 0$, we have:

$$\begin{aligned} [(ad + bc)/bd] &= [(a'd' + b'c')/b'd'] \Leftrightarrow \\ (ad + bc)/bd &\sim a'd' + b'c')/b'd' \Leftrightarrow \\ adb'd' + bcb'd' &= bda'd' + bdb'c' \Leftrightarrow \\ bcb'd' &= bdb'c' \Leftrightarrow \\ cd' &= dc' \Leftrightarrow \\ c/d &\sim c'/d'. \end{aligned}$$

The last statement is equivalent to $[c/d] = [c'/d']$, and the proof is finished.

We call an equivalence class of $\sim$ a **rational number**, or just a **rational** and denote the set of all rational numbers by $\mathbb{Q}$.

The operations of addition and multiplication of rationals defined above satisfy the following properties. We will write xy instead of $x \cdot y$.

Proposition 1.1. *For all rational numbers x, y, z, the following hold.*

(1) *Commutativity:* $x + y = y + x$, *and* $xy = yx$.
(2) *Associativity:* $(x + y) + z = x + (y + z)$, *and* $(xy)z = x(yz)$.
(3) *Identity elements:* $[0/1] + x = x + [0/1] = x$ *and* $[1/1]x = x[1/1] = x$, *and* $[0/1]$ *and* $[1/1]$ *are the only rationals with these properties.*
(4) *Inverses: for a given rational x, there is a unique rational u such that* $x + u = u + x = [0/1]$;
for a given rational $x \neq [0/1]$, there exists a unique rational v such that $xv = vx = [1/1]$.
(5) *Distributivity:* $x(y + z) = xy + xz$ *and* $(x + y)z = xz + yz$

The rational u in part (4) is denoted by $-x$ and is called the **additive inverse** of x, or the **opposite** of x. The rational v in part (4) is denoted by x^{-1} and is called the **multiplicative inverse** of x, or the **reciprocal** of x.

Proof. We prove only two of these statements, and leave others as exercises.

(1) Let $x = [a/b]$ and $y = [c/d]$. Then $x + y = [(ad + bc)/(bd)]$. Using the commutative properties of addition and multiplication of integers, we rewrite $ad+bc$ as $cb + da$, and bd as db. Hence,

$$x + y = [(ad + bc)/(bd) = [(cb + da))/(db)] = y + x.$$

Similarly for multiplication:

$$xy = [(ac)/(bd)] = [(ca)/(db)] = yx.$$

(4) We prove it for the multiplicative inverse only. For the additive inverse the argument is similar.

Let $x = [p/q] \neq [0/1]$. Then $p \neq 0$, and we can consider $v = [q/p]$. So we have:

$$xv = [(pq)/(qp)] = [(pq)/(pq)] = [1/1].$$

As $xv = vx$ by (1), we also have $vx = [1/1]$. This shows that v is a multiplicative inverse of x. Why v is unique? Let $v'x = xv' = [1/1]$. Then, using the associativity of the multiplication of rationals (which is left for the reader to check),

$$v' = v' \cdot [1/1] = v'(xv) = (v'x)v = [1/1] \cdot v = v,$$

hence, $v' = v$. □

As an immediate corollary of Proposition 1.1 we obtain several more useful properties of operations on $\mathbb{Q}$. They all can be proved as *logical* consequences of statements (1) – (5) of Proposition 1.1, i.e., without any reference to the nature of elements of $\mathbb{Q}$.

Proposition 1.2. *For all rational numbers x, y, z, the following hold.*

(1) $x + y = x + z$ *if and only if* $y = z$.
(2) *If* $xz = yz$ *and* $z \neq [0/1]$, *then* $x = y$.
(3) $[0/1] \cdot x = x \cdot [0/1] = [0/1]$.
(4) $xy = [0/1]$ *if and only if* $x = [0, 1]$ *or* $y = [0/1]$.
(5) $(-x)y = -(xy) = x(-y)$, $-x = (-1)x$, $(-x)(-y) = xy$.
(6) *If* $x, y \neq 0$, *then* $(xy)^{-1} = x^{-1}y^{-1}$, *and* $(x^{-1})^{-1} = x$.

Proof. Let us prove (2), (3), and the first statement of (5). Proofs of other statements are left as exercises.

(2) As $z \neq 0$, there exists z^{-1}. Then, as $xz = yz$, $(xz)z^{-1} = (yz)z^{-1}$. Using parts (2) and (4) of Proposition 1.1, we obtain:

$(xz)z^{-1} = x(zz^{-1}) = x \cdot [1/1] = x$, and $(yz)z^{-1} = y(zz^{-1}) = y \cdot [1/1] = y$. This implies $x = y$.

(3) We have: $[0/1] \cdot x = ([0/1] + [0/1]) \cdot x = [0/1] \cdot x + [0/1] \cdot x$. Adding $-([0/1] \cdot x)$ to both sides and using (1), we get $[0/1] = [0/1] \cdot x$. As $[0/1] \cdot x = x \cdot [0/1]$, the statement is proven.

(5) Using the distributive property (5) of Proposition 1.1, and part (3), we obtain

$$xy + (-x)y = (x + (-x))y = [0/1] \cdot y = [0/1].$$

So $xy + (-x)y = [0/1]$, which means that $(-x)y$ is the additive inverse of xy, namely $-(xy)$, or simply $-xy$. □

1.2. **Are integers rational numbers?** Strictly speaking: no. What is true, that a function

$$f : \mathbb{Z} \to \mathbb{Q},$$

defined by $a \mapsto [a/1]$, is an injection satisfying the following properties:

$$f(a + a') = [(a + a')/1] = [a/1] + [a'/1] = f(a) + f(a'),$$

and

$$f(aa') = [(aa')/1] = [a/1] \cdot [a'/1] = f(a) \cdot f(a').$$

The properties show that the symbols $f(a) = [a/1]$, when it comes to operations, just mimic the ones in $\mathbb{Z}$, and differ from them only in appearance. This is often phrased by saying that the substructure $\{[a/1] : a \in \mathbb{Z}\}$ of $\mathbb{Q}$ is **isomorphic** to $\mathbb{Z}$.

1.3. **Let's make things simpler.** Now, when we know what rational numbers are, we will begin to denote them by fractions, as we used to. So, when we write $1/2 \in \mathbb{Q}$, we really mean $[1/2] \in \mathbb{Q}$. We also remember that equivalent fractions denote the same rational number, and from now on, writing $1/2 = 4/8$ is O.K. And, yes, writing a instead of $[a/1]$, or writing $\mathbb{Z} \subset \mathbb{Q}$, is fine too. Hence, $[0/1]$ can be written as just 0, and $[1/1]$ as just 1. What a relief!

Well, our time was not exactly wasted: now we *know* what this all means!!! Don't you feel like Monsieur Jourdain, who suddenly discovered that "*These forty years now, I've been speaking in prose without knowing it! How grateful am I to you for teaching me that!*"[2]?

1.4. **Order on $\mathbb{Q}$.** We remind the reader that we assume known the properties of the (usual) order relation $\leq$ on $\mathbb{Z}$. Similar properties of the order on $\mathbb{Q}$, which we are going to introduce, will follow from them easily.

We say that $x = a/b \in \mathbb{Q}$ is **positive** (**negative**), and write $0 < x$ ($x < 0$), if and only if ab is a positive (**negative**) integer.[3]

The sum $x + (-y)$ is called the **difference** of x and y, and it is usually denoted by $x - y$.

For $x, y \in \mathbb{Q}$, we say that x is **less** than y, and write it as $x < y$, if $y - x$ is positive. The inequality $x < y$ is often expressed in another way by saying that y is **greater** than x, and by using the symbol $>$ to denote it. Hence $y > x$ is equivalent to $x < y$.[4]

The abbreviation for "$(x < y)$ or $(x = y)$" is $x \leq y$. If $x \leq y$, we say that x is **at most** y, or, equivalently, y is **at least** x. Similarly, for $x \geq y$. Hence, both $3 \leq 3$ and $3 \leq 5$ are true. If $0 \leq x$ or, equivalently, $x \geq 0$, we say that x is **non-negative**, and if $x \leq 0$ or, equivalently, $0 \geq x$, we say that x is **non-positive**.

[2]From J.B.P. Moliére: *Le Bourgeois Gentilhomme*, 1670.

[3]If one wants to be more pedantic, and check that the notion is well-defined for rationals, one can observe that if $a/b \sim c/d$, then ab and cd are both positive or both negative simultaneously. Indeed, $a/b \sim c/d$ implies $ad = bc$, which is equivalent to $abd^2 = b^2cd$ (as $b, d \neq 0$). As b^2 and d^2 are positive, ab and cd are both positive, or both negative.

[4]This is an example of a wasteful expansion of mathematical vocabulary which is hard to justify. Maybe, saying "x is less than y" makes us to concentrate on x more than on y, while saying "y is greater than x" does otherwise? Same also applies to the next paragraph.

Proposition 1.3. *Suppose x, y, z represent rational numbers. Then the following holds.*

(1) *If $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$*
(2) *If $x < 0$ and $y < 0$, then $x + y < 0$ and $xy > 0$.*
(3) *If one of x, y is positive and another is negative, then $xy < 0$.*
(4) $1 > 0$.
(5) $x > 0$ $(x < 0)$ *if and only if* $-x < 0$ $(-x > 0)$.
(6) $x > 0$ *($x < 0$) if and only if* $x^{-1} > 0$ *($x < 0$).*
(7) *Only one of the following three statements is true for a number x:*

$$x = 0, \quad x > 0, \quad x < 0.$$

(8) $x \leq x$ *($x \geq x$)*
(9) *If $x \leq y$ and $y \leq x$, then $x = y$.*
(10) *If $x \leq y$ ($x \geq y$) and $y \leq z$ ($y \geq z$), then $x \leq z$ ($x \geq z$).*

Proof. We prove only three of these statements, and leave others as exercises.

(1) Let $x = a/b$ and $y = c/d$. Then both ab and cd are positive integers. As $x+y = (ad+bc)/bd$, we consider $(ad+bc)(bd) = (ad)(bd)+(bc)(bd) = (ab)d^2+b^2(cd)$. We know that integers ab, b^2, cd and d^2 are all positive, and sums and products of positive integers are positive. So we conclude that $(ad + bc)(bd)$ is positive, which is equivalent to $x + y > 0$. Similarly, $xy = (ac)/(bd)$, and $(ac)(bd) = (ab)(cd) > 0$ as the product of two positive integers. Hence, $xy > 0$.

(4) Since $1 = 1/1$, and $1 \cdot 1 = 1$, and 1 is a positive integer, then 1 (now viewed as rational) is a positive rational.

(10) Let us prove the version involving $\geq$. Since $x \geq y$ and $y \geq z$, both $x - y$ and $y - z$ are non-negative rationals by definition. Due to (1), which we proved above, the sum of positive rational is positive. This implies that the sum of two non-negative numbers is non-negative. So $(x - y) + (y - z) = x - z \geq 0$, hence, $x \geq z$. □

The order relation and the operations on $\mathbb{Q}$ satisfy the following additional properties. The expression $0 < x < y$ represents, of course, a shorthand of the statement: "$0 < x$ and $x < y$".

Proposition 1.4. *For any rational numbers x, y, z, a, b, the following holds.*

(1) $x < y$ *if and only if* $x + z < y + z$
(2) *For $z > 0$, $x < y$ if and only if $xz < yz$*
(3) *For $z < 0$, $x < y$ if and only if $xz > yz$*
(4) *If $a < b$ and $x < y$, then $a + x < b + y$, and a similar statement holds for any $n \geq 2$ inequalities.*
(5) *If $0 < a < b$ and $0 < x < y$, then $ax < by$, and a similar statement holds for any $n \geq 2$ inequalities*
(6) *The inequalities $0 < a < b$ imply $0 < a^n < b^n$, for any $n \in \mathbb{N}$. Also, if $b > 0$, and $n \in \mathbb{N}$, then $a^n < b^n$ implies $a < b$.*

Proof. Here we present proofs of several of the properties, leaving proofs of the remaining ones to the reader. We wish to note that these proofs will not be based on our understanding of what rationals are: they are merely logical consequences of the properties stated in Propositions 1.1, 1.2, 1.3.

(2) The distributive law of Proposition 1.1 (5) holds also when addition is replace by subtraction. Indeed, applying Proposition 1.2 (5), we obtain:

$$yz - xz = yz + (-xz) = yz + (-x)z = (y + (-x))z = (y - x)z.$$

If $x < y$, then $y - x > 0$ is positive by definition. Since $z > 0$ (given), and the product of positive rationals is positive by Proposition 1.3 (1), $(y - x)z$ is positive. Hence $yz - xz > 0$, and so $xz < yz$.

Conversely: let $xz < yz$. Then $yz - xz = (y - x)z > 0$ by definition. As $z > 0$, this implies $y - x > 0$, otherwise $(y - x)z < 0$ by Proposition 1.3 (3), or $y = x$. Hence, $x < y$.

(5) *Proof 1.* By transitivity of inequalities, $b > 0$. By part (2) of this proposition proven above, $a < b$ implies $ax < bx$, and $x < y$ implies $xb < yb$. By the commutativity of multiplication of rationals, the latter is equivalent to $bx < by$. Transitivity of inequalities gives $ax < by$.

Proof 2. $by - ax = by - bx + bx - ax = b(y - x) + (b - a)x$. As all $b, y - x, b - a$, and x are positive, $by - ax$ is positive. Hence, $ax < by$.

Let us generalize this property:

For all $n \geq 2$, the inequalities $0 < a_i < b_i$, $i = 1, \ldots, n$, imply
$$a_1 a_2 \cdots a_n < b_1 b_2 \cdots b_n.$$

We prove this by the method of mathematical induction (on n). The base case, $n = 2$, has been established in (5): $a_1 = a, b_1 = b$, $a_2 = x, b_2 = y$. Suppose the statement holds for any $n = k \geq 2$ inequalities. We have to show that it holds for any $n = k + 1$ inequalities.

Let $0 < x_i < y_i$ for all $i = 1, \ldots, k + 1$. We have to show that

$$x_1 x_2 \cdots x_k x_{k+1} < y_1 y_2 \cdots y_k y_{k+1}.$$

Note that $x_1 \cdots x_k < y_1 \cdots y_k$ by the induction hypothesis, and we are given that $x_{k+1} < y_{k+1}$. As all parts of the inequalities are positive rationals, these two inequalities can be multiplied (base case $n = 2$). By doing this, we obtain

$$(x_1 \cdots x_k)x_{k+1} < (y_1 \cdots y_k)y_{k+1}.$$

When several rational numbers are multiplied, the parentheses can be placed in arbitrary way due to the associative property of multiplication. Hence, the statement holds for any $n = k + 1$ inequalities, and the proof is finished. □

All properties of the order on $\mathbb{Q}$ we have seen so far are exactly the same as the one's on $\mathbb{Z}$. But there are some fundamental distinctions. For example, unlike for integers, the order relation on $\mathbb{Q}$ satisfies the following 'density' property.

Proposition 1.5. (Density property of $\mathbb{Q}$) *Between any two distinct rationals there are infinitely many other rationals.*

Proof. Suppose x, y be distinct rationals, and $x < y$. Let $z_1 = (x + y)/2$. Then $z_1 \in \mathbb{Q}$,

$$z_1 - x = \frac{x + y}{2} - x = \frac{y - x}{2} > 0, \text{ and}$$
$$z_1 - y = \frac{x + y}{2} - y = \frac{x - y}{2} < 0.$$

Hence, $x < z_1 < y$. Repeating the argument for x and z_1, we set $z_2 = (x + z_1)/2$. Then z_2 is a rational number, and $x < z_2 < z_1$. Continuing this way we obtain an infinite decreasing sequence of rationals $z_{n+1} = (x + z_n)/2$, such that $x < z_n < y$ for all $n \geq 1$. □

On the other hand, an analog of the Well-ordering of integers does not hold in $\mathbb{Q}$. Indeed, for $a \in \mathbb{Q}$, let $\mathbb{Q}_{>a} = (a, b) := \{x \in \mathbb{Q} : a < x < b\}$. Then not every non-empty subset S of (a, b) contains the smallest element. For example, take $S = (a, c)$, with $a < c < b$. If $m \in S$ were the least element of S, then, by the density of $\mathbb{Q}$ (Proposition 1.5), the interval (a, m) would contain infinitely many rationals. Each of these rationals is in S and less than m, a contradiction.

1.5. **Representing rationals as decimal fractions.** It is well known that every rational number can be written as a decimal fraction, and that the "long division" algorithm can be used to do it. Have you ever seen the details? If not, here they are.

First we accept for granted that integers can be written "in base 10". E.g., the decimal 2304 stands for the number

$$2 \cdot 10^3 + 3 \cdot 10^2 + 0 \cdot 10^1 + 4 \cdot 10^0.$$

This implies that fractions with denominators 10, 100, 1000, etc., can also be written "in base 10", e.g.,

$$\frac{519006}{10000} = \frac{519006}{10^4} = \frac{5 \cdot 10^5 + 1 \cdot 10^4 + 9 \cdot 10^3 + 0 \cdot 10^2 + 0 \cdot 10^1 + 6 \cdot 10^0}{10^4} =$$

$$5 \cdot 10^1 + 1 \cdot 10^0 + 9 \cdot \frac{1}{10^1} + 0 \cdot \frac{1}{10^2} + 0 \cdot \frac{1}{10^3} + 6 \cdot \frac{1}{10^4}.$$

This, as we know, can also be shortened to a decimal 51.9006. Let us now review how we write fractions with denominators different than powers of 10 as decimals (finite or infinite). What is presented below is the "long division " algorithm, which most of us were taught in the middle school, by using a shorthand, like

$$14\overline{)123}.$$

Let $p/q \in \mathbb{Q}$, $q \geq 1$. Dividing p by q with remainder we obtain $p = aq + r_1$, where $0 \leq r_1 < q$. This gives

$$\frac{p}{q} = a + \frac{r_1}{q}.$$

If $r_1 = 0$, we write $p/q = a$, or $p/q = a.000\ldots$. If $r_1 \geq 1$, we divide $10r_1$ by q with remainder. We get $10r_1 = b_1 q + r_2$, where $0 \leq r_2 < q$. Since $0 \leq 10r_1 \leq 9q$, then $0 \leq b_1 \leq 9$. This gives

$$\frac{p}{q} = a + \frac{r_1}{q} = a + \frac{10r_1}{q} \cdot \frac{1}{10} = a + \frac{b_1 q + r_2}{q} \cdot \frac{1}{10} =$$

$$a + \left(b_1 + \frac{r_2}{q}\right) \cdot \frac{1}{10} = \left(a + b_1 \cdot \frac{1}{10}\right) + \frac{r_2}{q} \cdot \frac{1}{10} =$$

$$\left(a + b_1 \cdot \frac{1}{10}\right) + \frac{10r_2}{q} \cdot \frac{1}{10^2}.$$

If $r_2 = 0$, we write $p/q = a.b_1$, or $p/q = a.b_1000\ldots$. If $r_2 \geq 1$, we divide $10r_2$ by q with remainder. We get $10r_2 = b_2q + r_3$, where $0 \leq r_3 < q$ and $0 \leq b_2 \leq 9$. If $r_3 = 0$, we wrote $p/q = a.b_1b_2$, or $p/q = a.b_1b_2000\ldots$. If $r_3 \geq 1$, we have

$$\frac{p}{q} = \Big(a + b_1 \cdot \frac{1}{10} + b_2 \cdot \frac{1}{10^2}\Big) + \frac{r_3}{q} \cdot \frac{1}{10^2}.$$

Continuing this way we obtain

$$p/q = \Big(a + b_1 \cdot \frac{1}{10} + b_2 \cdot \frac{1}{10^2} + \cdots + b_n \cdot \frac{1}{10^n}\Big) + \frac{r_{n+1}}{q} \cdot \frac{1}{10^n} =$$

$$a.b_1b_2\ldots b_n + \frac{r_{n+1}}{q} \cdot \frac{1}{10^n},$$

and so forth. This leads to the following representation of p/q:

$$p/q = a.b_1b_2\ldots b_nb_{n+1}\ldots\ , \tag{1}$$

with $0 \leq b_i \leq 9$ for all i. If $r_k = 0$ for all $k > n$, we abbreviate $a.b_1b_2\ldots b_n000\ldots$ as $a.b_1b_2\ldots b_n$, and write $p/q = a.b_1b_2\ldots b_n$. In this case we often say that p/q is represented as a **finite decimal**. If $r_k \neq 0$ for infinitely many k, we obtain an **infinite decimal**.

The equality (1) may prompt a question: we understand completely its left side, but do we really have complete information about its right side? It will take too long to finish the writing of an infinite decimal... It turns out that the sequence $\{b_n\}_{n\geq 1}$ is actually rather simple, and it allows a short description. Let us explain it.

Writing rational numbers as decimals by using the algorithms above, often referred to as the **long division** algorithm, it is easy to notice that we always obtain a decimal with a repeating group of consecutive digits. For example,

- $23/6 = 3.8333\ldots$: digit 3 repeats beginning from the second decimal position;
- $1/11 = 0.09090909090909090909\ldots$, sequence 09 repeats beginning from the first decimal position;
- $3991/990 = 4.031313131\ldots$, sequence 31 repeats beginning from the second decimal positions.

Often one expresses this repetition phenomena by writing $23/6 = 3.8(3)$, $1/11 = 0.(09)$, $3991/990 = 4.0(31)$. Note that the same repeating pattern can be expressed in other forms: $23/6 = 3.8(33)$, $1/11 = 0.(090909)$, $3991/990 = 4.031(3131313131)$, but usually we use the shortest repeating sequence of decimals at the the position it begins. Let us show that the repetition always takes place, and that the shortest repeating sequence for a fraction p/q, $q \geq 2$, has length at most $q - 1$.

Theorem 1.6. *Let $p/q = a.b_1b_2\ldots b_nb_{n+1}\ldots$, $q \geq 2$. Then there exist positive integers i and j, $i < j$, such that*

$$p/q = a.b_1b_2\ldots b_i(b_{i+1}\ldots b_j),$$

with $j - i \leq q - 1$.

Proof. Look again at the long division algorithm. If $r_i = 0$ for some i, then the statement is proven, as $i + 1 = j$ and $b_{i+1} = b_j = 0$. If $r_i \neq 0$ for all i, then, as there are at most $q-1$ distinct nonzero remainders when an integer is divided by q, two of the remainders must repeat. Hence, there exist indices i, j, $i < j$, such that

$r_i = r_j$. Then the ratios and the remainders of the divisions of $10r_i$ and $10r_j$ by q will be equal, i.e., $b_i = b_j$ and $r_{i+1} = r_{j+1}$. This, in turn, leads to $b_{i+1} = b_{j+1}$ and $r_{i+2} = r_{j+2}$, and so on. Hence, the repetition of a finite sequence of consecutive digits in the decimal representation of $p/q$is inevitable. Let j be the smallest index greater than i such that $r_i = r_j$. Then the remainders $r_i, r_{i+1}, \ldots, r_{j-1}$ are all distinct, and there are $(j-1)-i+1 = j-i$ of them. The remainders are elements of $\{1, 2, \ldots, q-1\}$. Hence, $j - i \leq q - 1$. □

Theorem 1.6 helps to speed computation of b_n for large n. Suppose we wish to know b_{100005} in

$$554867/26000 = 21.3410(384615).$$

Then b_{100005} is one of the six digits in the repeating sequence $3, 8, 4, 6, 1, 5$. As the repeating begins with $b_5 = 3$, we divide $100005 - 4$ by 6 with remainder, obtaining $100001 = 16666 \cdot 6 + 5$. Hence b_{100005} is the fifth member of the sequence $3, 8, 4, 6, 1, 5$, i.e., $b_{100005} = 1$.

1.6. **There is no rational number whose square is 2.** This was proven by the Pythagoreans (or maybe even by Pythagoras himself) in the 6th century BC, and is often considered as one of those "great moments" in mathematics which strongly affected its development.

Due to the Pythagorean Theorem for a right triangle, the length d of the diagonal of the unit square satisfies $d^2 = 1^2 + 1^2 = 2$. Pythagorean believed that integers rule the world, and everything could be expressed as certain relations among natural numbers: the motion of planets, physics, geometry, music ... In particular, it is believed that they believed (see Grattan-Guinness [8]) that the length of every segment can be expressed as a positive integer or as a ratio of integers. So one can imagine how shaken were they after *proving* that this is not the case! We wish to present and compare several different proofs of this result. For more proofs, see Bogomolny [4]. See also [22] and Henderson [10] for related discussions.

Proposition 1.7. *There is no rational number whose square is* 2.

Proof 1. This proof is, probably, the the most popular one. It assumes two facts:

- no odd integer is equal to an even integer, and
- every fraction can be written in its "lowest terms", i.e., with having its numerator and denominator relatively prime. (The latter follows from the Well-ordering of integers).

Suppose the contrary, and let $(m/n)^2 = 2$ with integers m, n being relatively prime. Then $m^2 = 2n^2$.

We claim that m is even. Indeed, if m were odd, then it could be written as $m = 2s+1$, for some integer s, and this would lead to $m^2 = 4s^2+4s+1 = 2(2s^2+2s)+1$– an odd number. The obtained contradiction implies that m is even. Hence, $m = 2s$ for some integer s. Then, $(2s)^2 = 2n^2$, or $2s^2 = n^2$. Arguing as before, we obtain that n is even. Having both m and n even, contradicts the assumption that they are relatively prime. This ends the proof. □

Proof 2. This proof assumes that

- 2 is a prime number,
- every non-prime integer can be written as a product of prime numbers, and the number of primes in this product is uniquely defined.

(So it uses a little less than the uniqueness of prime factorization).

Suppose the contrary, and let $(m/n)^2 = 2$. Then $m^2 = 2n^2$. A square of any integer (≥ 2) has an even number of prime factors in its prime factorization, namely twice as many as the number itself. Therefore, as 2 is prime, the equal numbers m^2 and $2n^2$ have, respectively, even and odd number of primes in their prime factorizations. A contradiction. □

Proof 3. This proof uses the Well-ordering of integers and simple inequalities. Let $(m/n)^2 = 2$. We may assume that both m and n are positive integers, otherwise we replace them with their absolute values. Since the set of m's in such rationals m/n is not empty, there must be one with the smallest m. Since $1 < m/n < 2$, we have $0 < 2n - m < m$ and $0 < m - n < n$. As

$$\frac{m}{n} = \frac{2n-m}{m-n}, \text{ we obtain } \left(\frac{2n-m}{m-n}\right)^2 = 2,$$

a contradiction, since $0 < 2n - m < m$. □

Proof 4. The following argument is a geometric version of Proof 3, though it may take a few minutes to see it. It deals with a geometric analog of a common divisor of two integers. It is very much in the spirit of Euclid's *Elements*, and, possibly, it is close to one of the oldest proofs of this result. See Rademacher and Töplitz [18], and Courant and Robbins [5] for related discussions.

Two line segments $\overline{AB}$ and $\overline{CD}$ are called **commensurable** if and only if there exists some third segment $\overline{EF}$ that can be laid end-to-end a whole number, say m, of times to produce a segment congruent to AB, and also, with a different whole number, say n, a segment congruent to CD. In this case $\overline{EF}$ is called a **common measure** of $\overline{AB}$ and $\overline{CD}$. In terms of the lengths of the segments, it means that $AB = m \cdot EF$ and $CD = n \cdot EF$.

It is easy to argue that two segments are commensurable if and only if the ratio of their lengths is a rational number. Indeed, let AB and CD be commensurable. Then $AB = m \cdot EF$, $CD = n \cdot EF$, and AB/CD is a rational number m/n. Conversely, let the ratio of lengths of line segments AB and CD be rational, say p/q. Divide $\overline{CD}$ into q congruent segments, and call one of them $\overline{EF}$. Then $\overline{EF}$, laid end-to-end q times gives $\overline{CD}$, and p times gives $\overline{AB}$. Hence, segments $\overline{AB}$ and $\overline{CD}$ are commensurable.

If a leg of a right isosceles triangle is commensurable with its hypothenuse, then the equation $x^2 = 2$ has rational solutions. Indeed, let a be the length of the leg. Then the length of the hypothenuse is $\frac{m}{n}a$, for $m/n \in \mathbb{Q}$, and

$$\left(\frac{m}{n}a\right)^2 = a^2 + a^2 \quad \Leftrightarrow \quad \left(\frac{m}{n}\right)^2 = 2.$$

In what follows we show that the leg and the hypothenuse are *not* commensurable. This will prove that the equation $x^2 = 2$ has *no* rational solutions.

Let $\overline{CA}$ be a leg of an isosceles right triangle ABC with the hypothenuse $\overline{AB}$, see Figure 1. Then $CA < AB < CA + CB = 2 \cdot CA$. Let C' be a point on $\overline{AB}$ such that $BC' = BC$. Let $\overline{BB'}$ bisect $\angle B$, where B' is on $\overline{CA}$.

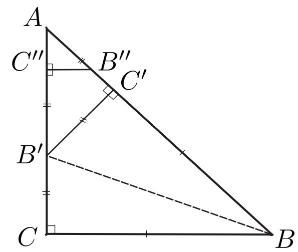

FIGURE 1.

As $\triangle B'CB$ is congruent to $\triangle B'C'B$ by Side-Angle-Side axiom, $\angle B'C'B$ is a right angle and $CB' = C'B'$. Hence, $\angle AC'B'$ is a right angle. As the measure of $\angle A$ is $45°$, $\triangle AC'B'$ is isosceles, so $AC' = B'C' = CB'$.

Suppose that $\overline{CA}$ and $\overline{AB}$ are commensurable, and $\overline{EF}$ is a common measure. It is clear that if two segments are commensurable, then the segment whose length is the difference of their lengths is commensurable with them. Hence, $\overline{EF}$ is also a common measure of $\overline{AC'}$ and $\overline{AB'}$. Thus we obtained a smaller right isosceles triangle $AC'B'$ such that $\overline{EF}$ is a common measure of its leg and its hypothenuse. Repeating the argument for this triangle, we obtain a smaller right isosceles triangle $AC''B''$ with $\overline{EF}$ being a common measure of its legs and its hypothenuse. Continuing this way, we obtain a decreasing (prove it) sequence of lengths

$$CA > C'A > C''A > \dots,$$

where each number is an integer multiple of EF. Writing $CA = m \cdot EF$, $C'A = m' \cdot EF$, $C''A = m'' \cdot EF$, etc., we get

$$m > m' > m'' > \dots,$$

which is an infinite decreasing sequence of positive integers. Its existence, clearly contradicts to the Well-ordering of integers. The obtained contradiction proves that $\overline{CA}$ and $\overline{AB}$ are not commensurable. □.

1.7. **An unsolved problem.** We finish this section by mentioning an open problem. It is related to representing rationals as decimals. Suppose p is a positive odd prime number. Considering representations of several fractions $1/p$ as decimals we have:

$$\begin{aligned} 1/3 &= 0.(3), \\ 1/7 &= 0.(142857), \\ 1/11 &= 0.(09), \\ 1/13 &= 0.(076923), \text{ and} \\ 1/17 &= 0.(0588235294117647). \end{aligned}$$

From these examples one can see that, when p is prime, the length of the shortest repeating sequence of decimal digits in the decimal expansion of $1/p$ can be $p-1$ $(p = 7, 17)$, or less $(p = 3, 11, 13)$. It is never greater than $p-1$ by Theorem 1. It is an not known whether there are infinitely many primes p for which this sequence contains exactly $p-1$ digits. For more on this, see [18].

Exercises for Section 1.

All problems below can be solved without any reference to real numbers. That is why we avoided geometric terminology which would make some statements more appealing. Solving problems below, one may assume, if needed, that a rational can be represented by a fraction p/q, where $\gcd(p,q) = 1$. By * we mark harder problems (in our opinion).

1.1 Prove the reflexive and symmetric properties of the relation $\sim$ on the set of fractions as defined in this section.

1.2 Show that the multiplication of rationals introduced in this section is well-defined.

1.3 Prove all properties of operations on rationals from Proposition 1.1.

1.4 Prove all properties of operations on rationals from Proposition 1.2.

1.5 Prove all properties of inequalities on rationals from Proposition 1.3.

1.6 Suppose a positive integer n is not a square of another integer. Prove that it is also not a square of a rational number.

1.7 Show that there is no rational number r such that

$$\text{(i) } r^5 = 96 \qquad\qquad \text{(ii) } 3^r = 7$$

1.8 A rational solution of the equation $x^2 + y^2 = 1$ is an ordered pairs (a, b) with both $a, b \in \mathbb{Q}$ and $a^2 + b^2 = 1$. Find ten distinct rational solutions of $x^2 + y^2 = 1$.

1.9 Check that for every rational number r, the rational numbers

$$x = (r^2 - 1)/(r^2 + 1) \text{ and } \quad y = (2r)/(r^2 + 1)$$

satisfy the equation $x^2 + y^2 = 1$.

This proves that the equation $x^2 + y^2 = 1$ has infinitely many rational solutions.

1.10 * Prove that the equation $x^2 + y^2 = 3$ has no rational solutions, i.e., there are no rational numbers x and y which satisfy it. This shows that not every rational number is a sum of squares of two rational numbers.

1.11 Check that

$$9m = \Big(m - 1\Big)^3 + \Big(\frac{3(m^2 + m)}{m^2 + m + 1}\Big)^3 + \Big(\frac{-m^3 + 3m + 1}{m^2 + m + 1}\Big)^3$$

Substituting $m = a/9$ gives a representation of any rational a as a sum of cubes of three rational numbers. Compare this result with the one of Problem 1.10.

Now, of course, one wants to know how such a representation can be found, and are there other representations. For this, see Shklarsky, Chentzov, Yaglom, Sussman,[20] (an elementary exposition), or Cohen [6] (for good college students), or Manin [14] (an advanced treatment).

1.12 Can the long division algorithm produce decimal
(a) $0.111\cdots = 0.(1)$?
(b) $0.131313\cdots = 0.(13)$?

(c) * $0.999\ldots = 0.(9)$?

Comment. One can show that if a decimal not of the form $0.b_1 \ldots b_k(9)$, it can always be obtained by applying the long division to certain integers.

1.13 Let $\alpha = p/q = a.b_1 b_2 \ldots b_n \ldots$.... Find b_{1000} if

(i) $\alpha = 0.22(345)$ (ii) $\alpha = 3991/990$.

1.14 (Rational root theorem) Let p/q, $\gcd(p,q) = 1$, be a rational solution of the equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \text{ with all } a_i \in \mathbb{Z},\ a_n \neq 0.$$

Then p divides a_0 and q divides a_n.

Comment. This fact is useful for finding all rational solutions of polynomial equations with integer coefficients (or showing that no such exist). It reduces the problem to examining only *finitely* many fractions p/q. The statement also implies that if $a_n = 1$, then every rational solution of the equation is actually an integer.

1.15 (i) Find all rational solutions of the equation

$$2x^5 - 9x^4 + 12x^3 - 12x^2 + 10x - 3 = 0.$$

(ii) Show that there is no rational solution of

$$x^5 - 7 = 0.$$

1.16 * Prove that for each rational number r such that $r^2 < 2$, there exists a rational number t, such that $r < t$ and $t^2 < 2$.

1.17 * Prove that $\mathbb{Q}$ is a countable set, i.e., there exists a bijection from $\mathbb{N}$ to $\mathbb{Q}$.

1.18 Is there a set $\{a_1, a_2, \ldots, a_k\}$ of rational numbers such that every rational number r can be written in the form $r = n_1 a_1 + n_2 a_2 + \ldots + n_k a_k$, for some integers n_i?

1.19 (a) Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that $f(a+b) = f(a) + f(b)$ for all $a, b \in \mathbb{Q}$. Prove that f is always of the form $f(x) = cx$ for for some $c \in \mathbb{Q}$ and all $x \in Q$. A function with this property is called an **additive** function on $\mathbb{Q}$.

(b) Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Q}$, and $f(1) = 1$. A function with this property is called a **multiplicative** function on $\mathbb{Q}$. Describe all multiplicative functions on $\mathbb{Q}$.

(c) Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function which is both additive and multiplicative on $\mathbb{Q}$. Prove that f is the identity function, i.e., $f(x) = x$ for all $x \in \mathbb{Q}$.

2. **Real numbers.**

2.1. **Why are they needed?** It is clear that for all everyday life purposes one does not need numbers other than rationals, especially when one uses decimal fractions. Every measurement is approximate, and finite decimals can express the result with needed precision. They are sufficient for building chairs, houses, and even rockets. But they are not sufficient for measuring the lengths of all segments! Well, segments are not objects encountered in everyday life... Still it is so restrictive not be able to say what is the length of the diagonal of a square with a unit side! An easily perceived correspondence between real numbers and points of a straight line made real numbers even more ‘real’, and lead first to the Coordinate method, and then to Calculus. These two theories proved to be extremely useful, both in mathematics and applications. That is why one needs reals.

2.2. **Definition of real numbers.** What is a real number? Most of the time, when one thinks about a real number, one imagines symbols like 0.234, or -21.998167, or 4532.12121212... which are called **decimals** or **decimal fractions**. Though decimals were used informally in our discussion of rationals, here we need to do it again, and at a slower pace. Decimals are infinite sequences made out of ten digits $0, 1, 2, \ldots, 9$ preceded with a symbol $+$ or $-$, and one symbol represented by a period “.” . (The last period is the grammatical symbol, not mathematical!) Usually, when terms of a sequence are arranged in a row, they are separated by commas. We do not do it in case of decimals.

A general form of a decimal is

$$\alpha = \pm a_m a_{m-1} \ldots a_1 a_0 . b_1 b_2 \ldots b_n \ldots, \text{ where all } a_i, b_j \in \{0, 1, 2, \ldots, 9\}.$$

The integers a_i and b_j are called the **decimal digits** of α. In such a representation, if $m \geq 1$, a_m is assumed to be non-zero. Any decimal of the form $0.00 \ldots 0 b_N b_{N+1} b_{N+2} \ldots$, $N \geq 1$, is called a decimal **tail** of α, or the Nth decimal tail of α.

To simplify appearance of decimals, several conventions are agreed upon. Usually the $+$ which precedes the decimal is not used: $0.2341984\ldots$ is just a shorthand for $+0.2341984\ldots$. Usually a tail with each term equal to 0 is dropped, and we write $0.234000\ldots$ as 0.234. The latter is an example of a **finite** decimal. So every finite decimal is a decimal with tail of all 0’s: $43.45 = 43.45000\ldots$. Usually the period is not used if it follows by a tail of all 0’s: $34.000\ldots$ is the same as 34, and -2.0 is the same as -2 . Another convention, which we will motivate later (see Subsection 2.4), is that any decimal with a tail of all 9’s is assumed to be equal to a finite decimal with the digit preceding the tail increased by 1:

$$\begin{aligned} 0.233999\ldots &= 0.234, \\ 1.999\ldots &= 2, \\ -1.999\ldots &= -2, \\ -3.24999\ldots &= -3.25\,. \end{aligned}$$

We call decimals described above **real** numbers and denote the set of all of them by $\mathbb{R}$. Often we refer to real numbers as to “reals”.

Having described the symbols we use to identify real numbers provides very little information about them. The most important property of objects referred to as numbers is that one can *compute* with them, i.e., to perform some operations, like addition or multiplication. Also, often one wants to be able to *order* them, and be able to say that one number is smaller or greater than another. For reals represented by decimals, the latter is much easier than the former. Indeed, everyone knows how to add or multiply two finite decimals, but how does one add two infinite decimals?

2.3. **Order on decimals.** Ordering reals is easy, and we all know how to do it:

$$\begin{aligned} 23.9891 &< 111.343434\ldots \\ 23.9891 &< 23.9991 \\ 23.9891 &< 23.989100001 \\ -5 &< 0.000001 \\ -5.00001 &< -5, \text{ etc.} \end{aligned}$$

The sign $<$ is read '**less**'. A general rule of ordering reals can be stated this way. Decimals which begin with $+$ (often omitted) are said to be **positive**, and those which begin with $-$, **negative**. We denote the set of all positive decimals by $\mathbb{R}^+$, and the set of all negative decimals by $\mathbb{R}^-$. We assume that the **zero** decimal $0 = 0.000\ldots$ is the only decimal which is neither positive nor negative. For any negative decimal α, and any positive decimal α', we declare $\alpha < 0$, $0 < \alpha'$, and $\alpha < \alpha'$. For any two distinct positive decimals

$$\alpha = a_m a_{m-1} \ldots a_1 a_0.b_1 b_2 \ldots b_n \ldots$$

and

$$\beta = c_s c_{s-1} \ldots c_1 c_0.d_1 d_2 \ldots d_n \ldots,$$

we declare

$$\alpha < \beta$$

if one of the following conditions is satisfied:

- $m < s$, or
- $m = s$ and, proceeding from left to right in both decimals and comparing the corresponding decimal digits one at a time, we obtain that for the first time they are distinct, the digit of α is less than the corresponding digit of β.

The second case, $m = s$, can be presented more formally in the following way.

(1) $m = s$, and there exists an integer k, $0 \le k \le m$, such that $a_m = c_m$, $a_{m-1} = c_{m-1}$, ... , $a_{k+1} = c_{k+1}$, but $a_k < c_k$, or
(2) $m = s$, and $a_i = c_i$ for $i = 0, 1, \ldots, m$, and there exist an integer $k \ge 1$, such that $b_1 = d_1$, $b_2 = d_2$, ... , $b_{k-1} = d_{k-1}$, but $b_k < d_k$.

Being able to compare any two positive decimals α and β, we compare $-\alpha$ and $-\beta$ in the reverse way: $-\alpha < -\beta$ if and only if $\beta < \alpha$. (This was a definition, by the way.)

The definitions of 'greater'($>$), 'at most' ($\le$), 'at least' ($\ge$), 'non-positive', and 'non-negative' are exactly the same as for the rationals. One can also easily check that all the properties of the order on rationals listed in parts (7) - (10) of Proposition 1.3 are satisfied by the just introduced order on reals.

We also have an analog of the density property of rationals:

Proposition 2.1. (Density property of $\mathbb{R}$) *Between any two distinct reals, there are infinitely many other reals.*

Note that at this place of our discussion we cannot prove this statement in the way we did it for rationals in Proposition 1.5, as we do not know yet how to add reals. Nevertheless, a proof based on ordering of reals, can be easily given. The following numerical example illustrates the idea of such a proof. A general argument is similar, and we omit it.

Let $\alpha = 23.25983\ldots$ and $\beta = 23.25984\ldots$. Then $\alpha < \beta$, as no tail of α consists of 9's only. The latter also implies that there are infinitely many decimal digits in the 6-tail of α which are less than 9. Increasing any one of them by 1, and not changing other decimal digits, we obtain infinitely many decimals which are greater than α and less than β.

2.4. **Forbidding tails of all 9's.** Forbidding tails of all 9's, allows us to say that if reals "look" distinct, they are distinct. Otherwise, if we considered $1.999\ldots$ and $2.000\ldots$ as distinct reals, we would not be able to find any other real between them, and the density property will fail. When reals are matched with points of a line, this would introduce a 'gap' on the latter which our intuition resists to accept. A few other reasons will be explained later.

2.5. **Periodic decimals.** A decimal $\alpha = \pm a_m a_{m-1} \ldots a_1 a_0 . b_1 b_2 \ldots b_n \ldots,$ is called **periodic**, if there exist a positive integer N and a positive integer p such that for all $n \geq N$, $b_{n+p} = b_n$:

$$\exists N \in \mathbb{N} \;\; \exists p \in \mathbb{N} \;\; \forall n \geq N \; (b_{n+p} = b_n).$$

If a decimal is periodic, numbers N and p are not defined uniquely. For example, $12.333\ldots$ (all decimal digits are 3) is periodic, with $N = 1, p = 1$, or with $N = 3$ and $p = 5$. Decimal $12.1341341341\ldots$ (sequence of digits 341 repeats) is periodic, with $N = 2, p = 3$, or with $N = 4$ and $p = 6$.

If α is periodic, p is called a **period** of α. It is clear that if p is a period of α, then kp is also a period for all integers $k \geq 0$. Indeed, for $n \geq N$,

$$b_{n+2p} = b_{(n+p)+p} = b_{n+p} = b_n,$$

so $2p$ is a period. Similarly, for $n \geq N$,

$$b_{n+3p} = b_{(n+2p)+p} = b_{n+2p} = b_n,$$

so $3p$ is a period, etc..

It can be shown that if a decimal is periodic, then there exists the smallest period, and it is called the **principal** period. For example, if

$$\alpha = 4.031313131\ldots,$$

then periods have length 2, 4, 6, …, the value of the principal period is 2, and α is often written as $4.0(31)$, or $4.0(\overline{31})$.

Here are some not periodic decimals:

$$\beta = 0.101001000100001000001\ldots$$

(runs of 0's increase in length and are separated by 1's;)

$$\gamma = 0.12345678910111213\ldots$$

(consequent natural numbers written next to each other;)

Let us explain why β is not periodic. If it is, then there exist positive integers N and p such that $b_{n+p} = b_n$ for all $n \geq N$. As the lengths of sequences of consecutive 0's grow, there exists a positive integer $M \geq N$ such that

$$b_M = b_{M+1} = \ldots = b_{M+p-1} = 0.$$

Since p is a period, so are $2p, 3p, 4p$, etc., then

$$b_M = b_{M+1} = \ldots = b_{M+p-1} =$$

$$b_{M+p} = b_{M+1+p} = \ldots = b_{M+2p-1} =$$

$$b_{M+2p} = b_{M+1+2p} = \ldots = b_{M+3p-1} = \cdots = 0,$$

i.e. β is periodic with a tail of all 0's. This certainly contradicts the definition of β. Hence, it is not periodic.

The non-periodicity of γ can be shown in a similar way.

2.6. **Continuity of reals.** Soon we will see that reals posses all good properties of rationals. Here we discuss a fundamental property of real numbers which rationals do not possess, the property of continuity, which makes reals superior to rationals when it comes to Calculus.

Let $S \subseteq \mathbb{R}$. A number $A \in \mathbb{R}$ is called an **upper bound** of S if every element of S is at most A:

$$A \text{ is an upper bound of } S \text{ if } \ \forall x \in S \ (x \leq A).$$

A subset of reals which has an upper bound is called **bounded from above**. For example, $(-\infty, 1)$ is bounded from above by any number greater or equal to 1.

An upper bound α of $S \subset \mathbb{R}$ is called a **least upper bound** of S if it is not greater than any upper bound of S. If $S \subseteq \mathbb{R}$ has a least upper bound, it is, clearly, unique, so we can speak about *the* least upper bound.

The least upper bound of S is often called the **supremum** of S, and it is denoted $\sup(S)$. For example,

$$\sup((-\infty, 1)) = \sup((-\infty, 1]) = \sup((0, 1)) = 1.$$

Similar definitions can be made of a **lower bound** of S, of a set **bounded from below**, and of the **greatest lower bound** of S. The greatest lower bound of S is often called the **infimum** of S, and it is denoted by $\inf(S)$.

Theorem 2.2. (Continuity of Real Numbers.) *Every nonempty bounded from above (below) subset of real numbers has the unique least upper (greatest lower) bound in* $\mathbb{R}$.

Before proving the theorem, we wish to explain that it does not hold for rational numbers, more precisely, that not every bounded from above a nonempty subset of $\mathbb{Q}$ has a supremum in $\mathbb{Q}$.

Proposition 2.3. *Let* $S = \{r \in \mathbb{Q} : r^2 < 2\}$. *Then* S *is a non-empty bounded from above in* $\mathbb{Q}$ *and has no supremum in* $\mathbb{Q}$.

Proof. As $1 \in S$, S is nonempty. Obviously, 2 is an upper bound of S, so S is bounded from above. Suppose $\alpha = \sup(S)$, where $\alpha \in \mathbb{Q}$. We show that this assumption leads to a contradiction. Note that (if exists) $\alpha > 1$.

As $\alpha^2 = 2$ is not possible by Proposition 1.7, we have $\alpha^2 < 2$ or $\alpha^2 > 2$.

Case 1: $\alpha^2 < 2$. Then $\alpha \in S$. Consider any rational t such that

$$0 < t < \min\{1, (2-\alpha^2)/(2\alpha+1)\}.$$

Then

$$(\alpha+t)^2 = \alpha^2 + 2\alpha t + t^2 = \alpha^2 + (2\alpha + t)t < \alpha^2 + (2\alpha+1)t <$$
$$\alpha^2 + (2\alpha+1)\frac{2-\alpha^2}{2\alpha+1} = \alpha^2 + (2-\alpha^2) = 2.$$

Hence, $(\alpha+t)^2 < 2$, and so $\alpha + t \in S$. As $\alpha < \alpha + t$, α is not an upper bound of S, a contradiction.[5]

Case 2: $\alpha^2 > 2$. Then $\alpha \notin S$ and α is an upper bound of S. Consider any rational t such that

$$0 < t < \min\{1, (2-\alpha^2)/(-2\alpha+1)\}.$$

Then

$$(\alpha-t)^2 = \alpha^2 - 2\alpha t + t^2 = \alpha^2 + (-2\alpha + t)t > \alpha^2 + (-2\alpha+1)t >$$
$$\alpha^2 + (-2\alpha+1)\frac{2-\alpha^2}{-2\alpha+1} = \alpha^2 + (2-\alpha^2) = 2.$$

Hence, $(\alpha-t)^2 > 2$, and so $\alpha - t$ is a positive upper bound of S smaller than α, a contradiction.

Therefore there is no supremum of S in $\mathbb{Q}$. □

We now proceed with a proof of Theorem 2.2. Let $\lfloor x \rfloor$ denote the integer part of x, defined as the greatest integer less or equal x. For example, $\lfloor 6.23 \rfloor = \lfloor 6 \rfloor = 6$, and $\lfloor -6.23 \rfloor = \lfloor -7 \rfloor = -7$.

Proof. Let $\emptyset \neq S \subseteq \mathbb{R}$.

Case 1. S contains nonnegative numbers.

Let $S_0 = \{\lfloor x \rfloor : x \in S\}$. Then S_0 is a non-empty subset of integers. As S is bounded from above, so is S_0. By the Well-Ordering Axiom of integers, S_0 contains the greatest element.[6] Denote it by a. Then $a \geq 0$. Consider a set S_1 obtained from S by removing from it all numbers with integer part less than a: $S_1 = \{x \in S : \lfloor x \rfloor = a\}$. As $a \in S_1$, S_1 is not empty. Each element $x \in S_1$ can be represented as a decimal $x = \overline{a}.x_1x_2x_3\ldots$, where $\overline{a}$ denote the representation of a in base 10. Let $b_1 = \max\{x_1 : x = \overline{a}.x_1x_2x_3\ldots \in S_1\}$. Removing from S_1 all numbers x with $x_1 < b_1$, we obtain a nonempty subset of $S_2 = \{x \in S_1 : x = \overline{a}.b_1x_2x_3\ldots\}$. Let $b_2 = \max\{x_2 : x = \overline{a}.b_1x_2x_3\ldots \in S_2\}$. Let $S_3 = \{x \in S_2 : x = \overline{a}.b_1b_2x_3\ldots\}$. Continuing this way we construct a decimal $\alpha = \overline{a}.b_1b_2b_3\ldots b_n\ldots$, which is not

[5] All inequalities in this argument are applied to rational numbers only, and hold due to Propositions 1.3 and 1.4.

[6] Often the Well-Ordering Axiom is presented in the following version: every nonempty subset X of integers bounded from below contains the least element. This is equivalent to the the statement that every nonempty subset X of integers bounded from above contains the greatest element. To see the equivalence, consider the set $-X = \{-x : x \in X\}$.

necessarily in S. Clearly, α is an upper bound of S. We claim that $\alpha = \sup(S)$. Let $\gamma = \overline{c}.d_1d_2d_3\ldots d_n\ldots$ (with no tail of all 9's) is an upper bound of S. We have to show that $\alpha \le \gamma$.

Suppose α has no tail of all 9's. If $\alpha = \gamma$, we are done. Suppose $\alpha > \gamma$. Then either $a > c$, or $a = c \ge 0$ and there exists an index $n \ge 1$ such that $b_n > d_n$ and $b_i = d_i$ for indices smaller than n (if any). If $a > c$, then c is not an upper bound of S, a contradiction. In the latter case, S contains a number of the form $x = \overline{a}.b_1b_2b_3\ldots b_n\ldots \ge \overline{a}.b_1b_2b_3\ldots b_n > \gamma$, a contradiction again.

It may happen that α has a tail of all 9's. As no element of S has such a tail, this will happen only if $\alpha = \overline{a}.(9)$, or for some $k \ge 2$, S contains all numbers of the form $\overline{a}.b_1\ldots b_{k-1}9$, $\overline{a}.b_1\ldots b_{k-1}99$, $\overline{a}.b_1\ldots b_{k-1}999,\ldots$, where $b_{k-1} < 9$. In the first case, we rewrite α as $a+1$, and in the second – as $\overline{a}.b_1\ldots b'_{k-1}$, with $b'_{k-1} = b_{k-1}+1$. As in the previous case, S again contains a number of the form $x = \overline{a}.b_1b_2b_3\ldots b_n\ldots \ge \overline{a}.b_1b_2b_3\ldots b_n > \gamma$, a contradiction.

Hence, $\alpha = \sup(S)$.

Case 2. S contains only negative numbers.

Consider a number $\delta > 0$ such that the set $S+\delta := \{x+\delta : x \in S\}$ contains nonnegative reals. Clearly such a δ exists. Then $\sup(S+\delta)$ exists, as we just proved in Case 1. Let's call it α. Then, obviously, $\alpha - \delta = \sup(S)$.

To prove the theorem with respect to infimum, replace a nonempty bounded from below set S of reals by $-S = \{-x : x \in S\}$. As $-S$ is bounded from above, by the argument above, it has the least upper bound in $\mathbb{R}$, say α. Then, obviously, $-\alpha = \inf(S)$. □

It seems that a logical way to finish this subsection is to show that there exists a real number whose square is 2. This would partially justify our efforts of introducing real numbers. Rereading our argument which preceded the proof of Theorem 2.2, makes it clear that the supremum of S in $\mathbb{R}$ must have this property. The only problem with this "clear" is that at this time we do not know yet that the laws of operations and inequalities for reals are similar to the ones for rationals. Moreover, we have not introduced any operations on $\mathbb{R}$ yet! Therefore, we have to address this issue first.

2.7. **Computations with decimals.** In this section we use the fact that finite decimal fraction can be identified with (some) rational numbers, and that usual operations on them have the properties listed in Proposition 1.1, except the existence of multiplicative inverses for each nonzero number. (though $x = 3 = 3.(0)$ is a finite decimal, $x^{-1} = 0.(3)$ is not.)

Let $\alpha = \pm a_ma_{m-1}\ldots a_1a_0.b_1b_2\ldots b_n\ldots$. For each $n \ge 1$, we define a finite decimal

$$\begin{aligned}\alpha_n &= \pm a_ma_{m-1}\ldots a_1a_0.b_1b_2\ldots b_n000000\ldots\\ &= \pm a_ma_{m-1}\ldots a_1a_0.b_1b_2\ldots b_n\end{aligned}$$

Then α_n is called the n**th approximation** of α. Let β be another decimal, and let β_n be its nth approximation. Counting with finite decimal fractions in the usual way, we define two new sequences of finite decimals: for each integer n, $n \ge 1$, let

$$\sigma_n := \alpha_n + \beta_n, \quad \text{and} \quad \rho_n := \alpha_n \cdot \beta_n.$$

Then, one can prove (but we do not do it here) that there exists a unique decimal σ, and a unique decimal ρ such that, for every $n \geq 1$, σ_n and ρ_n are the nth approximations of σ and ρ, respectively. The decimals σ and ρ are called the **sum** and the **product** of α and β, respectively, and we write $\sigma = \alpha+\beta$ and $\rho = \alpha\beta$ (or $\alpha \cdot \beta$). Here is an example. Let

$$\alpha = 2.398745012\ldots \quad \text{and} \quad \beta = 5.137096541\ldots$$

Then

$$\begin{array}{lcl}
\alpha_1 + \beta_1 = & 2.3 + 5.1 & = 7.4 \\
\alpha_2 + \beta_2 = & 2.39 + 5.13 & = 7.52 \\
\alpha_3 + \beta_3 = & 2.398 + 5.137 & = 7.535 \\
\alpha_4 + \beta_4 = & 2.3987 + 5.1370 & = 7.5357 \\
\alpha_5 + \beta_5 = & 2.39874 + 5.13709 & = 7.53583 \\
\alpha_6 + \beta_6 = & \ldots + \ldots & = 7.535841 \\
\alpha_7 + \beta_7 = & \ldots + \ldots & = 7.5358415 \\
\alpha_8 + \beta_8 = & \ldots + \ldots & = 7.53584155 \\
\alpha_9 + \beta_9 = & \ldots + \ldots & = 7.535841553
\end{array}$$

and

$$\begin{array}{lcl}
\alpha_1\beta_1 = & 2.3 \cdot 5.1 & = 11.73 \\
\alpha_2\beta_2 = & 2.39 \cdot 5.13 & = 12.2607 \\
\alpha_3\beta_3 = & 2.398 \cdot 5.137 & = 12.318526 \\
\alpha_4\beta_4 = & 2.3987 \cdot 5.1370 & = 12.32212190 \\
\alpha_5\beta_5 = & 2.39874 \cdot 5.13709 & = 12.3225432666 \\
\alpha_6\beta_6 = & \ldots \cdot \ldots & = 12.322583344520 \\
\alpha_7\beta_7 = & \ldots \cdot \ldots & = 12.32258454389250 \\
\alpha_8\beta_8 = & \ldots \cdot \ldots & = 12.3225846912132654 \\
\alpha_9\beta_9 = & \ldots \cdot \ldots & = 12.322584703886203492
\end{array}$$

Hence $\sigma = 7.53584155\ldots$ and $\rho = 12.322584\ldots$. Let us explain why we do know the first six decimal digits of ρ. To get an upper bound of ρ, we can just multiply two numbers slightly greater than the nth approximations of α and β for any fixed n. Taking $n = 8$, we obtain ,

$$12.3225846912132654 < \rho < (\alpha_8 + 10^{-8})(\beta_8 + 10^{-8}) =$$

$$2.39874502 \cdot 5.1370966 = 12.322584886508932.$$

Hence, $12.3225846 < \rho < 12.3225848$. Therefore $\rho = 12.322584\ldots$.

One can show that the opposite of α is the same decimal as α but with the altered sign. It is denoted by $-\alpha$. One can also show that the difference $\alpha - \beta$, defined as $\alpha + (-\beta)$, is positive if and only if $\alpha > \beta$.

It turns out, but we omit the proofs, that the order, and the operations on reals, in addition to all properties of Propositions 1.3, 2.1 and Theorem 2.2, also satisfy all properties of Propositions 1.1, 1.2 and 1.4.

Now it is a good time accept the fact that

There exists a positive real number x whose square is 2.

It is denoted by $\sqrt{2}$. The argument preceding the proof of Theorem 2.2 from subsection 2.6, with the set S considered now as a subset of reals, provides a proof. It shows that $\sqrt{2} = \sup(S)$, which exists in $\mathbb{R}$.

By a similar argument, one can prove that $\sqrt{\alpha}$ exists for every real $\alpha \geq 0$.

2.8. **Where are we?** In the last section we used phrases like 'one can prove', or 'one can show', or 'it turns out ... , but we omit the proofs'. In case the reader is getting tired of these statements, we can assure the reader, that if we decided to present all these proofs, she or he would be even more tired ... Our goal here is not to present a self-contained mathematically rigorous exposition of a theory of real numbers. This is done in undergraduate courses very rarely, and rarely all the properties are verified even in graduate courses. The main reason is that every existing thorough exposition is long.[7]

Our goal in Section 2 was to review basic properties of reals, to mention some less known ones, and to point at the difficulties with the development of the theory. We also wanted to provide a model of real numbers. Instead of spending time on verifying all the details related to the model, we prefer to concentrate on other questions related to reals which we find more interesting. Let us mention them. If some of the notions or the properties in the list below are not familiar to the reader, keep reading and attempt solving problems. These notions and properties will play little role in the remaining part of these notes.

[7]A relatively fast and rigorous way of introducing reals is defining them axiomatically, e.g., as an ordered field with an additional Completeness axiom. Corresponding definitions and exposition can be found, e.g., in Birkhoff and MacLane [3] or D'Angelo and West [1]. In this approach many propositions/theorems become axioms, and other properties are derived from them. One can easily proceed with building new theories which use reals: limits of functions, Calculus, Topology, Geometry, etc. . The main weakness of this approach is that it does not provide any explicit description of reals in terms of other mathematical objects known to the reader. Such a description is often called a *model.* A model is necessary to ensure that the axiomatic system is consistent in the sense that no two contradictory statements can be proven in it. The model of decimals that we provided is, probably, the simplest. It was suggested by K. Weierstrass (1815-1897) in the second half of the 19th century. Weierstrass never published his theory of real numbers, so it is hard to date it. But notes of his lectures, taken at different years by several of his students, contain it.

Among other models of reals, we wish to mention the ones constructed by G. Cantor (1871) and by R. Dedekind (1856, 1872). For other constructions, see the article "Construction of the real numbers" on Wikipedia.org. For each construction one has to carefully check that all axioms of reals are satisfied. After this is done, we say that the construction is a *model* of reals. It is possible to show (see, e.g., [3]) that all model of reals are essentially the same (isomorphic), which is not a property most axiomatic system possess. The readers who knows basics of abstract algebra, can easily see that none of the axiomatic system for groups, or for rings, or for fields has the property that all corresponding models are isomorphic.

We take for granted the following additional notions and facts about reals:

- Limits and their properties
- Series and their properties (like convergence,...). In particular, that

$$0.b_1b_2\dots b_n\dots = \sum_{i=1}^{\infty}\frac{b_i}{10^i}.$$

- Continuity of elementary functions and related properties. These imply, e.g., that numbers $\sqrt[10]{1.2}$, $\sin 3$, $\arctan 2$, ... exist.
- The set of all periodic decimals can be identified with $\mathbb{Q}$, and operations are consistent with the ones in $\mathbb{R}$. In other words, we assume that the set of all periodic decimals is isomorphic to $\mathbb{Q}$, and we will write $\mathbb{Q} \subset \mathbb{R}$.
- $\mathbb{Q}$ is a countable set. $\mathbb{R}$ is an uncountable set.

We have seen that the long division algorithm applied to a fraction always leads to a periodic decimal (Theorem 1.6). On the other hand, it is possible to argue that for every periodic decimal α (no tail of all 9's), there exist a rational number p/q such that the long division of p by q leads to α. We do not prove this fact, but illustrate two practical ways of doing it.

Example 1. Let $\alpha = 2.(7)$. Then (believe us!) $10\alpha = 27.(7)$, and (believe us again!) $10\alpha - \alpha = 9\alpha = 25$. Hence, $\alpha = 25/9$. If one doubts the method, one can just apply long division to 25 and 9. The answer will be 2.(7).

If the reader is familiar with infinite geometric series, then the same result can be obtained in the following way. First we take for granted that

$$2.(7) = 2 + \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \cdots = 2 + \sum_{n=1}^{\infty}\frac{7}{10^n}.$$

Then, using the formula for the sum of the infinite geometric progression, we obtain:

$$2.(7) = 2 + \sum_{n=1}^{\infty}\frac{7}{10^n} = 2 + \frac{7/10}{1-1/10} = 2 + 7/9 = 25/9.$$

Example 2. Let $\alpha = 5.3(71)$. Then $10\alpha = 53.(71)$ and $1000\alpha = 5371.(71)$. Subtracting, we get $990\alpha = 5318$. Hence $\alpha = 5318/990 = 2659/495$. Again, to check the result, one can apply long division to 2659/495. The answer will be 5.3(71).

Using infinite geometric progression, the same result can be obtained this way.

$$5.3(71) = 5 + \frac{3}{10} + + \frac{71}{10^3} + \frac{71}{10^5} + \frac{71}{10^7} + \cdots = \frac{53}{10} + \frac{71/10^3}{1-1/10^2} = \frac{53}{10} + \frac{73}{990} = \frac{2659}{495}.$$

2.9. **Irrationals.** Thus we assume that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R},$$

and that rational numbers correspond to precisely the periodic decimals. We have also shown that some decimals are not periodic, so they do not correspond to rational numbers. Reals which are not rational are called **irrational**. They form the set $\mathbb{R} \setminus \mathbb{Q}$. They correspond to (infinite) non-periodic decimals. Since $\mathbb{Q}$ is a countable set, and $\mathbb{R}$ is uncountable, and the union of two countable sets is countable, the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational real numbers is uncountable. Hence, the are "much more" irrational numbers than there are rationals.

When we deal with irrational numbers, in most cases we do not have them represented as non-periodic decimals. We describe them in some other way, like $\sqrt{2}$, or $\log_2 3$, $\cos 1$. Descriptions like these are useful if we wish to know which decimal digit occupies, say the 1000th decimal place, since, in these cases, there are relatively fast algorithms of computing the nth decimal digit. At the same time, no one knows what patterns are present in their representations as decimals. For example, no one knows whether there exists a sequence of 100 consecutive 5's in the decimal representing $\sqrt{2}$.

The questions whether given real numbers are rational or irrational have always occupied mathematicians. Some of them are relatively easy, others are very hard. It is hard not to like them. Often these are asked about numbers with simple definitions, and which often appear in mathematical reasonings: like

$$e,\ \pi,\ \log_2 3,\ \cos 1,$$

$$\frac{\arccos(1/3)}{\pi},$$

$$\zeta(3) := \sum_{i=0}^{\infty} \frac{1}{n^3},$$

$$\zeta(5) := \sum_{i=0}^{\infty} \frac{1}{n^5},$$

the Euler-Mascheroni constant

$$\gamma := \lim_{n\to\infty} \Big(\sum_{i=0}^{n} \frac{1}{i} - \ln n \Big).$$

For example, no one knows whether $\zeta(5)$ or γ is irrational. More on this subject can be found in the following books: Niven [15, 16], Schoenberg [19], Hardy and Wright [9], Khinchin [11], and Baker [2].

In the next section we consider several relatively simple problems of this kind.

2.10. **Rational or irrational?** As we know, the set $\mathbb{Q}$ of rationals is closed with respect to arithmetic operations, like the sum, difference, product, division (if the divisor is not zero). This means that the result of these operations is again a rational number. This is not true for irrationals. For example, the sum or the product of two irrationals can be a rational number.

On the other hand, a sum of a rational number and an irrational number is always irrational. Indeed, if the sum were rational, then the irrational number would be the difference between two rationals, a contradiction. See more in Problem 5.

Example 3. Prove that if $n \in \mathbb{N}$ and n is not a square of an integer, then $\sqrt{n}$ is irrational.

Proof. As $\sqrt{n}$ exists, it is either rational or irrational. Suppose it is rational. Then $\sqrt{n} = a/b$, for some $a, b \in \mathbb{N}$. Then $a^2 = nb^2$. As $n > 1$ and is not a square of an integer, the exponent of some prime divisor p of n in the prime decomposition of n is odd. As the exponents of p in the prime factorizations of a^2 and of b^2 are even, the exponent is even in a^2 and odd in nb^2. This contradicts the uniqueness of prime factorization. Hence $\sqrt{n}$ is irrational. □

Example 4. Prove that $\sqrt{2} + 3\sqrt{5}$ is irrational.

Proof. We would like to discuss three proofs of this statement. They illustrate different ideas.

Proof 1. Suppose $r = \sqrt{2} + 3\sqrt{5}$ is rational. Then $r^2 = 47 + 6\sqrt{6}$ is rational, and $\sqrt{6} = (r^2 - 47)/6$ is rational. As 6 is not a perfect square of an integer (Example 3), $\sqrt{6}$ is irrational, a contradiction. Hence, r is irrational.

Proof 2. Let $a_1 = \sqrt{2} + 3\sqrt{5}$, $a_2 = \sqrt{2} - 3\sqrt{5}$, $a_3 = -\sqrt{2} + 3\sqrt{5}$, and $a_4 = -\sqrt{2} - 3\sqrt{5}$. Consider the polynomial

$$f(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) =$$

$$(x^2 - (a_1 + a_2)x + a_1a_2)(x^2 - (a_3 + a_4)x + a_3a_4) =$$

$$(x^2 - 2\sqrt{2}x - 43)(x^2 + 2\sqrt{2}x - 43) =$$

$$(x^2 - 43)^2 - (2\sqrt{2}x)^2 = x^4 - 94x^2 + 43^2.$$

By the Rational root theorem (see Problem 14), the only possible rational roots of f can be 1, -1, 43, -43, 43^2 and -43^2. One can easily eliminate each of them as a root (by a direct substitution, for example). Hence, all four real roots of f, namely, all a_i are irrational. The argument demonstrates the irrationality of all four roots simultaneously. It also demonstrates that introducing "more symmetry" into the problem can be useful. This last observation was extremely important in the development of algebra in the last two centuries.

It is possible to argue, that f is a polynomial of the smallest degree with integer coefficients which has a_1 as its root. We do not do it here.

Proof 3. The polynomial f from the previous proof could also be found in the following way. Let $a = \sqrt{2}+3\sqrt{5}$. Then $(a-\sqrt{2})^2 = (3\sqrt{5})^2$, or $a^2-2\sqrt{2}a+2 = 45$. From here we have, $(a^2 - 43)^2 = (2\sqrt{2}a)^2$, or $a^4 - 94a^2 + 43^2 = 0$. Hence, a is a root of f. Now we apply the Rational root theorem, as in Proof 2. □

2.11. **Approximating some irrational numbers by rationals. Theorems of Kronecker and Dirichlet.** The fact that for any real number α there exist rational numbers arbitrarily close to it is obvious: just take the nth approximation α_n of α. The denominator of α_n is 10^n, and the approximation we get is within $10^{-(n+1)}$. Hence, in this case the distance to a number is no greater than a tenth of the denominator of the approximating fraction. An interesting question is how close one can get to α by using rational numbers with denominators at most b. We begin our discussion with the following result, which, I think, is very surprising at the first glance.

Example 5. Surround each rational number $\frac{a}{b} \in (1,2)$, $\gcd(a,b) = 1$, with an open interval $I_{a/b}$ which is centered at a/b and of radius $\frac{1}{3.5b^2}$, i.e.,

$$I_{\frac{a}{b}} = \Big(\frac{a}{b} - \frac{1}{3.5b^2}, \frac{a}{b} + \frac{1}{3.5b^2}\Big).$$

Does the union of all such intervals cover the interval $(1,2)$? In other words, is it true that

$$(1,2) \subseteq \bigcup_{\frac{a}{b}\in(1,2)} I_{\frac{a}{b}} \ ?$$

Solution. We claim that the answer is No. As every rational number is covered, we should try to find an irrational number which is not covered. Let us prove that $\sqrt{2}$ is one of such numbers.

Proof. Number $\sqrt{2}$ is not in the union of all $I_{a/b}$ if and only if it does not belong to any of them. In terms of inequalities it means that for every $\frac{a}{b} \in (1,2)$,

$$\Big|\sqrt{2} - \frac{a}{b}\Big| \geq \frac{1}{3.5b^2},$$

or, equivalently, that the inequality

$$-\frac{1}{3.5b^2} < \sqrt{2} - \frac{a}{b} < \frac{1}{3.5b^2}$$

does not hold for any rational a/b.

In order to prove this inequality, we first observe that $|2b^2 - a^2| \geq 1$ for all integer $a, b \geq 1$. We can assume $a, b \geq 1$ as $a/b > 0$. This implies

$$\begin{aligned}
|b\sqrt{2} - a||b\sqrt{2} + a| \geq 1 \quad &\Leftrightarrow \\
|b\sqrt{2} - a| \geq \frac{1}{|b\sqrt{2} + a|} \quad &\Rightarrow \quad (\text{as } 1 < \frac{a}{b} < 2) \\
\Big|\sqrt{2} - \frac{a}{b}\Big| > \frac{1}{b^2(\sqrt{2} + 2)} \quad &\Rightarrow \\
\Big|\sqrt{2} - \frac{a}{b}\Big| > \frac{1}{3.5b^2}.&
\end{aligned}$$

□

Having proved the statement, let us now try to understand it better. Two decimals are close to each other if their decimal expansions begin with many equal corresponding digits. We know that any rational number a/b can be presented as a decimal with a period no longer than $|b| - 1$. At the same time, an irrational number can be thought as a decimal with an infinite period. Therefore for the numbers to be close, $|b|$ must be large. When $|b|$ is large, the radius of $I_{a/b}$, namely $\frac{1}{3.5b^2}$, is small. The example above demonstrates that it is small enough for $I_{a/b}$ never 'capture' the irrational $\sqrt{2}$.

The result in Example 5 should be compared with the fact that there exist infinitely many rationals a/b which still come close to $\sqrt{2}$ in the sense that

$$-\frac{1}{b^2} < \sqrt{2} - \frac{a}{b} < \frac{1}{b^2}.$$

This is a particular case of the following beautiful theorem of Dirichlet (1842):

Theorem 2.4. (Dirichlet's Theorem.) *For any irrational number α there exist infinitely many rational numbers a/b such that*

$$-\frac{1}{b^2} < \alpha - \frac{a}{b} < \frac{1}{b^2}.$$

In light of Example 5 and this theorem, a question one may ask is of determining the *maximum* positive constant c such that the inequality

$$-\frac{1}{cb^2} < \sqrt{2} - \frac{a}{b} < \frac{1}{cb^2}$$

holds for infinitely many rationals a/b. Theorem 2.4 implies that this maximum value is at least 1, and, from Example 5 it is at most 3.5. It turns out that it is $2\sqrt{2}$ (for details, see Waldschmidt [21]).

Now we discuss another striking result related to distribution of certain numbers among reals. It has many surprising applications. In particular, the Dirichlet's Theorem 2.4 will follow from it.

We say that a set S of real numbers is **dense** in $\mathbb{R}$ if every open interval $(a, b) \subset \mathbb{R}$ contains a number from S. It is clear that if S is dense, then every interval actually contains infinitely many numbers from S.

Theorem 2.5. (Kronecker's Theorem.) *Prove that for any irrational number α, the set $K = \{n\alpha + m,\ m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.*

Proof. Note that both set $\mathbb{Z}$ and K is closed with respect to addition (and substraction) of its elements.

We remind the reader that for any real number x, $\lfloor x \rfloor$ denotes the integer part of x, defined as the greatest integer which does not exceed x. The fractional part of x is defined as $x - \lfloor x \rfloor$, and is denoted by $\{x\}$. For example, if $x = 2.12, 5, -3, -3.1$, then $\lfloor x \rfloor = 2, 5, -3, -4$, and $\{x\} = 0.12, 0, 0, 0.9$, respectively. Clearly, $\{x\} \in [0, 1)$ for all $x \in \mathbb{R}$.

Let us first show that K is dense in $(0, 1)$. The proof of the theorem will easily follow from this. For every $n \in \mathbb{N}$, let $x_n = \{n\alpha\}$.

Key observation 1. All x_n are distinct. Indeed, let $m \neq n$ and $x_n = x_m$.

This implies that

$$\{n\alpha\} = \{m\alpha\} \quad \Leftrightarrow$$
$$n\alpha - \lfloor n\alpha \rfloor = m\alpha - \lfloor m\alpha \rfloor \quad \Leftrightarrow$$
$$(n - m)\alpha = \lfloor n\alpha \rfloor - \lfloor m\alpha \rfloor.$$

As all $\lfloor n\alpha\rfloor, \lfloor m\alpha\rfloor, m-n$ are integers, and $m \neq n$, the last equality implies that α is rational, a contradiction. Hence all x_n are distinct.

Consider an interval $(a,b) \subset [0,1)$, and let $b-a=\epsilon$. We choose a positive integer N such that $1/N < \epsilon$, and divide $[0,1)$ into N subintervals of equal length:

$$\Big[0,\frac{1}{N}\Big),\quad \Big[\frac{1}{N},\frac{2}{N}\Big),\quad \ldots,\quad \Big[\frac{N-1}{N},1\Big).$$

Then at least two of $x_1, x_2, \ldots, x_{N+1}$ must be in the same subinterval. Suppose these are x_k and x_{k+h}, $h \geq 1$. Let $\delta = x_{k+h} - x_k > 0$ (if $\delta < 0$, the argument will continue in a similar way). Then $0 < \delta < 1/N < \epsilon$.

Key observation 2. Consider a subsequence (x_{k+th}), $t \geq 0$, of the sequence (x_n). If we bend $[0,1)$ into a circle, then points representing consecutive members of (x_{k+th}) will be spaced on the circle by the arcs of length δ.

Therefore, as all x_{k+th} are distinct, and the arc between the points representing a and b is of length $\epsilon > \delta$, at least one of x_{k+th} must be on the arc representing (a,b).[8] This completes the proof of the statement that K is dense in $[0,1)$.

Let us show that K is dense in $\mathbb{R}$. Let $(a,b) \subset \mathbb{R}$, $a < b$. We want to show that at least one point of K is in (a,b). Consider interval $(a-\lfloor a\rfloor, b-\lfloor a\rfloor)$. As $0 \leq a - \lfloor a\rfloor = \{a\} < 1$, we have $(a-\lfloor a\rfloor, b-\lfloor a\rfloor) \cap (0,1) = (\{a\}, c)$ where $c = \min\{b-\lfloor a\rfloor, 1\}$. As K is dense in $[0,1)$, there exist $s,t \in \mathbb{Z}$ such that $s\alpha + t \in (\{a\}, c)$. Then

$$a < (s\alpha + t) + \lfloor a\rfloor = s\alpha + (t + \lfloor a\rfloor) < b.$$

As $t + \lfloor a\rfloor \in \mathbb{Z}$, $(s\alpha+t) + \lfloor a\rfloor \in K \cap (a,b)$, and the proof is finished. □

Let us explain that the results obtained in the proof above imply the Dirichlet's theorem. We will use the notations and ideas from our proof of the Kronecker's theorem. There we have explained that for any $N > 1$, there are integers k and $k+h$, such that $1 \leq k < k+h \leq N+1$, and $|x_{k+h} - x_k| < 1/N$, which is equivalent to $|h\alpha - (\lfloor x_{k+h}\rfloor - \lfloor x_k\rfloor)| < 1/N$. Setting $b = h$ and $a = |\lfloor x_{k+h}\rfloor - \lfloor x_k\rfloor|$, we can rewrite the last inequality as $|b\alpha - a| < 1/N$, or $|\alpha - a/b| < 1/(bN)$. As $1 \leq b \leq N$, $bN \geq b^2$, and so $|\alpha - a/b| < 1/b^2$ for some $1 \leq b \leq N$. Thus we proved that for any positive integer $N > 1$ there exist *at least one* rational number a/b such that

$$|\alpha - \frac{a}{b}| < \frac{1}{bN} \leq \frac{1}{b^2} \quad \text{and} \quad 1 \leq b \leq N.$$

Note that the numbers a and b in this inequality can be assumed to be *coprime*. Indeed, if $d = \gcd(a,b) > 1$, and $a = pd$, $b = qd$, then

$$|\alpha - \frac{p}{q}| = |\alpha - \frac{a}{b}| < \frac{1}{bN} \leq \frac{1}{b^2} < \frac{1}{p^2} \quad \text{and} \quad 1 \leq p < b \leq N.$$

Now we explain that this implies that there are *infinitely many* rationals a/b such that $|\alpha - a/b| < 1/b^2$.

If the found a/b corresponded to $N > 1$, consider an integer N' such that $1/N' < |b\alpha - a| < 1/N$. Such N' exists since $|b\alpha - a| > 0$ due to the irrationality

[8] One can think of a point jumping around the circle with circumference length 1 in the same direction, each jump having length equal $\delta < \epsilon$. Then for any arc of this circle of length ϵ, at least one jump will land in it.

of α. As $N' > N$, $N' > 1$, and we know that there exist coprime integers a', b' such that $|b'\alpha - a'| < \frac{1}{N'}$ and $1 \le b' \le N'$. This implies that $|b'\alpha - a'| < |b\alpha - a|$. If $a/b = a'/b'$, then $a' = \pm a$ and $b' = \pm b$, as two fractions are in the lowest terms. As both b and b' are positive, $a' = a$ and $b' = b$, a contradiction with $|b'\alpha - a'| < |b\alpha - a|$. Hence $a'/b' \ne a/b$, and we have shown the existence of a new fraction a'/b' satisfying $|\alpha - a'/b'| < 1/(b')^2$. Applying similar argument to a'/b', we obtain a fraction a''/b'' distinct from a'/b' and satisfying the inequality. It is also distinct from a/b as it is in the lower terms and

$$|b''\alpha - a''| < |b'\alpha - a'| < |b\alpha - a|.$$

Now it is clear that continuing this way we can show that there are infinitely many distinct rationals a/b such that $|\alpha - a/b| < 1/b^2$. This ends the proof of Theorem 2.4. $\Box$

2.12. **Algebraic and transcendental numbers.** Some irrational numbers, like $\sqrt[5]{3}$, $1 - \sqrt{3}$, or $\sqrt{3} + \sqrt[5]{3}/2 - \sqrt[6]{2} + 51$, are solutions of polynomial equations with integer coefficients, i.e., of the the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \text{ all } a_i \in \mathbb{Z}, a_n \ne 0.$$

For the first number a polynomial can be $x^5 - 3 = 0$, for the second — $x^2 - 2x - 8 = 0$, and for the third — it can be worked out. Numbers of this type are called real **algebraic** numbers. Clearly, every rational is algebraic: $\frac{m}{n}$ is a root of $nx + m = 0$.

A real number which is not algebraic is called **transcendental**. The existence of transcendental numbers is not a trivial matter, but it follows easily from the facts that $\mathbb{R}$ is uncountable and the set of all algebraic numbers is countable (Cantor's argument, 1874). Therefore there are "much more" transcendental numbers than algebraic ones.

Liouville was the first who proved a theorem (in 1844) which enables us to produce examples of transcendental numbers. By using it, one can easily show that the number $\sum_{n=1}^{\infty} 10^{-n!}$ is transcendental.

It is much harder to prove that some particular well-known numbers are transcendental. It is known, for example, that e, π, e^{π}, $\log_2 3$, $\arcsin(1/3)$ are transcendental. And it is not known whether π^e, 2^e, $e + \pi$, $\ln(\ln 2)$ are transcendental. See [15], [2] , and [21] for much more on this subject.

Exercises for Section 2.

Assume that you know what is the field of real numbers $\mathbb{R}$, and its model via decimal fractions. If you need it, you can use as known that periodic decimals correspond precisely to the elements of $\mathbb{Q}$ and non-periodic decimals – to irrational numbers. We assume that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. In most problems below, one does not have to use the fact that reals can be represented by decimals. It is often useful to represent a non-zero rational by a fraction $\frac{m}{n}$, where $m, n \in \mathbb{Z}$, $n \neq 0$, and $\gcd(m, n) = 1$. We call a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ a **rational point** if all x_i are rational. Harder (in my opinion) problems are marked by *, and even harder – by **.

2.1. Thinking about reals as infinite decimal fractions, prove that between any two distinct real numbers there are infinitely many rational numbers and infinitely many irrational numbers.

2.2. (i) Prove that the decimal $.12345678910111213\ldots$ is not periodic.

(ii) Prove that the decimal $0.1001000010\ldots = \sum_{i=1}^{\infty} 10^{-n^2}$ is not periodic.

2.3. * Prove that for a periodic decimal, the principal period divides any other period.

2.4. Suppose n and k are positive integers, $k \geq 2$, and let n be not the kth power of another integer. Assuming that $\sqrt[k]{n}$ is real, prove that it is irrational.

2.5. Let $a, b \in \mathbb{R}$ and $a, b > 0$. What can be said (rational or irrational) about $a + b$, $a \cdot b$, and a^b if
(i) both a and b are rationals?
(ii) a is rational and b is irrational?
(iii) a is irrational and b is rational?
(iv) both a and b are irrationals?

2.6. There is no a standard notation for the set of all irrational reals. We have not introduce one either. Why, in your opinion, this is the case?

2.7. Is there a line in $\mathbb{R}^2$ passing through the origin and distinct from both x- and y-axes such that
(i) it contains no point distinct from the origin with both integer coordinates?
(ii) it contains no rational point distinct from the origin?
(iii) every point of the line distinct from the origin has exactly one rational coordinate?
(iv) it contains no point with both coordinates irrational?

2.8. In $\mathbb{R}^2$, consider the integer lattice $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$. Is there an equilateral triangle having three of the lattice as its vertices? Prove your answer.

2.9. In $\mathbb{R}^2$, consider the integer lattice $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$. Is it true that there exists a point in the plane which has different distances from all points of $\mathbb{Z}^2$?

2.10. In $\mathbb{R}^2$, consider the integer lattice $\mathbb{Z}^2 = \{(x,y) : x, y \in \mathbb{Z}\}$. Prove that for every $n \in \mathbb{N}$, there exists a circle in the plane having in its interior exactly n lattice points.

2.11. Consider Problems 9 and 10 for $n \geq 3$, i.e., for $\mathbb{R}^n$ and $\mathbb{Z}^n$.

2.12. $\sqrt[3]{2}$ cannot be represented as $a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$.

2.13. Determine whether the following real numbers are rational or irrational:
(a) $\sqrt{2} + \sqrt{3}$;
(b) $\sqrt{2} + \sqrt{3} + \sqrt{5}$;
(c) $\log_{10} 2$;
(d) * $\cos 1°$, $\sin 1°$;
(e) $\tan 1°$;
(f) * $\frac{1}{\pi} \arccos \frac{1}{3}$;
(g) $1 + 2^{1/3} + 2^{2/3}$;
(h) $1 + 3^{1/5} + 3^{2/5} + 3^{3/5} + 3^{4/5}$.

2.14. Determine whether the following real numbers are rational or irrational:
(a) $a_n = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}$ (n radicals), $n \geq 1$.
(b) $\alpha = \sqrt[3]{6 + \sqrt{847/27}} + \sqrt[3]{6 - \sqrt{847/27}}$

2.15. * Let $1 < p_1 < p_2 < \ldots < p_n$ be a sequence of primes. Is it true that $\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_n}$ is irrational? Prove your answer.

2.16. Prove that the first 999 decimal digits after the period in the decimal representation of $(6 + \sqrt{37})^{999}$ are all zeros.

2.17. Prove that if a line in $\mathbb{R}^2$ contains two rational points, then it contains infinitely many rational points.

2.18. Given a circle in $\mathbb{R}^2$. What is the greatest number of rational points can be on it if its center is not a rational point?

2.19. * Is there a circle in $\mathbb{R}^2$ with exactly 51 rational points on it? What about with exactly 56 rational points on it?

2.20. Consider the unit circle $S^1 : x^2 + y^2 = 1$, and a point $A : (-1, 0)$ on it.
(i) Let B be a rational point of S^1 distinct from A, and let $M : (0, r)$ be the intersection of line AB with the y-axis. Prove that $r \in \mathbb{Q}$.
(ii) Let $M : (0, r)$ where $r \in \mathbb{Q}$, and let B be the intersection of line AM with S^1. Then B is a rational point of S^1.
(iii) Parts (i) and (ii) can be combined in the following statement: there exists a bijective correspondence between $\mathbb{Q}$ and the set of all rational points of $S^1 \setminus \{A\}$. Prove the set of all rational points on S^1 coincides with the following set:

$$\{(-1,0)\} \cup \left\{ \left(\frac{1-r^2}{1+r^2}, \frac{2r}{1+r^2} \right) : r \in \mathbb{Q} \right\}.$$

We say that the set of all rational points of $S^1 \setminus \{A\}$ is **parametrized** by $\mathbb{Q}$, or that we have found a general form of rational points on S^1.

2.21. Using the ideas described in Problem 20, parametrize the set of all rational points of the ellipse $x^2/4 + y^2/9 = 1$.

2.22. Describe all rational numbers x for which $3x^2 - 5x + 9$ is a square of a rational number.

2.23. * Prove that there are no rational points on the circle $x^2 + y^2 = 3$.

2.24. Show that for any $n \in \mathbb{N}$, one can find n points in a plane, not all colinear, such that the distance between any two of them is an integer.

2.25. ** Suppose we have an infinite set of points in a plane with the integer distances between any two of the points. Prove that the points are colinear.

2.26. ** Can one find an infinite subset of rational points of the unit circle $x^2 + y^2 = 1$ such that all distances between them are also rational?

2.27. A man walks along the x-axis towards $+\infty$. The length of his step is an irrational number α. Each integer point n is surrounded with a "narrow ditch" of radius $\epsilon > 0$: $(n - \epsilon, n + \epsilon)$. Prove that sooner or later the man will step into the ditch.

2.28. Prove that for some $n \in \mathbb{N}$, the decimal representation of 2^n begins with 9999999999 (ten nines).

2.29. Assuming that π is irrational, prove that for some integer $n \neq 0$,

$$|\sin n| < 0.000000001.$$

2.30. In $\mathbb{R}^2$, consider the integer lattice $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$ and a line $y = mx$, where m is irrational. For a given $\epsilon > 0$, surround every point of the lattice with a disc centered at the point and of radius ϵ. Prove that no matter how small ϵ is, the line intersects infinitely many of such discs.

2.31. Prove that Theorem 2.4 does not hold for rational α.

2.32. * Let α and β be two positive irrational numbers such that $1/\alpha + 1/\beta = 1$. Let $A = \{\lfloor n\alpha \rfloor : n \in \mathbb{N}\}$ and $B = \{\lfloor n\beta \rfloor : n \in \mathbb{N}\}$. Prove that the sets A and B partition $\mathbb{N}$, which means that $A \cup B = \mathbb{N}$ and $A \cap B = \emptyset$.

2.33. (i)** Prove that for $n > 10$ there is no rational number m/n such that

$$-\frac{1}{n^3} < \sqrt{3} - \frac{m}{n} < \frac{1}{n^3}.$$

(ii) Find all rational numbers m/n which satisfy the inequality in part (i).

Hints and Answers to Some Exercises from Section 1

1.1 Use definitions and an example from the section.

1.2 Use definitions and an example from the section.

1.3 Use definitions and an example from the section.

1.4 Use the corresponding properties of integers.

1.5 Use definitions and an example from the section.

1.6 Hint: Use ideas of proofs in subsection 1.6. Or use Problem 1.14.

1.7 Hint: (i) Use ideas of proofs in subsection 1.6. Or use Problem 1.14. (ii) Hint: If $r = m/n$, then $3^m = 7^n$.

1.8 First find ten solutions of $x^2 + y^2 = z^2$ in integers. Can use computer, if you wish.

1.9 Just check.

1.10 Hint: first show that if $x \in \mathbb{Z}$, then $x^2 \equiv 0$ or $1 \bmod 3$. Then show that if $3|(x^2 + y^2)$ for $x, y \in \mathbb{Z}$, then both x and y are divisible by 3. Reduce the problem about rational solutions to the problem about integer solutions.

1.11 Just check.

1.12 (a) Yes (b) Yes (c) No.

1.13 (i) 4 (ii) 3 4.031313131...

1.14 Hint: Let p/q be a solution of the equation. We may assume $\gcd(p, q) = 1$. Substitute p/q into the equation, and get rid of the denominators. Proof of this fact can be found in many algebra books.

1.15 Answers: (i) $1, 1/2, 3$. (ii) No rational solutions.

1.16 Hint 1: Let $r = a/b \in \mathbb{Q}$. Check that $t = (2a + 2b)/(a + 2b)$ satisfies the required property. (I learned this idea from Elizabeth Sieminski (F.L.).)

Hint 2: Let $t = r + x$. Take a rational number x such that $0 < x < \min\{1, (2 - r^2)/(2r + 1)\}$. Prove that $t^2 < 2$. A well known argument; easy to locate.

1.17 Find a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. If you cannot do it, try to read about it in a book or on the web. Then find a bijection $f : \mathbb{N} \to \mathbb{Q}^+$, where $\mathbb{Q}^+$ denote the set of positive rational numbers. Then find a bijection $g : \mathbb{Q}^+ \to \mathbb{Q}$. The composition of f and g is an example of desired the desired bijection.

1.18 No.

1.19 (a) Hint: $f(1) = f(1/n + 1/n + \ldots + 1/n) = nf(1/n)$.

(b) Hint: What is $f(0)$? Is there any relation between $f(a)$ and $f(a^{-1}$, if $a \neq 0$? Think about the prime factorization of integers.

(c) Hint: Use parts (a) and (b).

Hints and Answers to Some Exercises from Section 2.

2.1. Try to use an idea from Section2. Or begin by showing that there are arbitrarily small positive rational and arbitrarily small positive irrational numbers.

2.2. Both (i) and (ii) are similar to a problem from Section2.

2.3. Suppose p is the principal period and p' is a period. Let $p' = qp + r$, where $0 \le r < p$. Show that if $0 < r$, then r is a period.

2.4. Hint: Find a similar example in the text.

2.5. Answers:
   (i) Rational; rational; either.
   (ii) Irrational; irrational; either.
   (iii) Irrational; irrational; either.
   (iv) Either; either; either.

2.6. The set of irrational numbers is not closed under arithmetic operations, like addition or multiplication. This makes it a much less interesting object from the algebraic point of you when they are compared to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

2.7. (i) Yes.
   (ii) Yes.
   (iii) No.
   (iv) No.

2.8. No. Use that $\tan 60° = \sqrt{3}$.

2.9. Yes, and $(\sqrt{2}, 1/3)$ is such a point. If the reader is familiar with the cardinalities of infinite sets, the existence also follows from the countability of perpendicular bisectors defined by all pairs of lattice points.

2.10. Follows from Problem 9.

2.11. For $n = 3$, one can consider a point $(\log 2/\log 7, \log 3/\log/7, \log 5/\log 7)$. Generalize. Another approach for those who are familiar with existence of transcendental numbers: consider a point $(\alpha, \alpha^2, \ldots, \alpha^n)$, where α is transcendental.

2.12. Assume the contrary.

2.13. (a) Irrational. Can use one of several ideas for a similar problem discussed in Section 2.
   (b) Irrational. Can use one of several ideas for a similar problem discussed in Section 2.
   (c) Irrational. Use uniqueness of prime factorization.
   (d) Both irrational. Show that $\cos n\alpha$, $n \in \mathbb{N}$, can be represented as a polynomial of $\cos\alpha$ with integer coefficients. Show that $\sin n\alpha$, $n \in \mathbb{N}$ and odd, can be represented as a polynomial of $\sin\alpha$ with integer coefficients.
   (e) Irrational. The ideas in the hint to the previous problem can be used, but there are also other ways.
   (f) Let $x = \cos^{-1}(1/3)$. Prove that for every positive integer n,
   $$\cos nx = \frac{a_n}{3^n},$$
   where a_n is an integer and $\gcd(a_n, 3) = 1$.
   (g) Irrational.
   (h) Irrational.

2.14. (a) Irrational. Hint: for $n \geq 2$, $a_n = \sqrt{2 + a_{n-1}}$.
(b) Rational. Hint: $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$.

2.15. Yes. Generalize and use induction on n.

2.16. First show that $(6+\sqrt{37})^{999} + (-6+\sqrt{37})^{999}$ is an integer. Then show that $0 < -6 + \sqrt{37} < 1/10$.

2.17. Consider an equation of the line.

2.18. At most two. Show that it is possible to have 0, or exactly 1, or exactly two rational points.

2.19. For 51 points, the answer is No. If there are more than 2 rational points, then the center is a rational point. Consider symmetries with respect to the center.

The answer to the second question is No, and the reason is deeper. Prove that if there are more than 2 rational points on the circle, then it contains infinitely many of them.

2.20. (i) Consider an equation of the line.
(ii) Consider an equation of the line.
(iii) There is nothing to add. Maybe this: does the result remind you a very useful formula from trigonometry? Expressing $\sin x$ and $\cos x$ in terms of $\tan(x/2)$? Can you explain the connection?

2.21. Either repeat the argument from Problem 20, or reduce the problem to Problem 20 by a simple change of variables.

2.22. Using the ideas described in Problem 20, parametrize the set of all rational points of the hyperbola $3x^2 - 5x + 9 = y^2$. Point $(0, 3)$ is on the curve.

2.23. First, explain that a remainder of the division of square of an integer by 3 can be only 0 or 1. Then explain that if the sum of squares of two integers is divisible by 3 then each of them is divisible by 3. Then assume that $x^2 + y^2 = 3$ has a rational solution, represent it as a pair of reduced fractions, and obtain a contradiction with the previous statement.

2.24. One can use Problem 20, for example.

2.25. Assume the contrary. Use the fact that in any triangle ABC, $|AB - AC| < BC$. Then show that the set can be represented as the union of points of intersection of finitely many hyperbolas, and that two distinct hyperbolas can intersect in at most 4 points.

2.26. Prove that there exists $\alpha > 0$ such that α/π is irrational, but both $\cos(\alpha/2)$ and $\sin(\alpha/2)$ are rational. Then consider $A = \{(\cos n\alpha, \sin n\alpha) : n \in \mathbb{N}\}$.

2.27. Use Theorem 2.5.

2.28. The decimal representation of 2^n begins with ten nines if and only if there exists a positive integer k such that

$$9999999999 \cdot 10^k < 2^n \leq 10^{k+10},$$

which is equivalent to

$$\log_{10}(0.9999999999) < n \log_{10} 2 - (k + 10) < 0.$$

Now use Theorem 2.5.

2.29. First show that $|\sin x - \sin y| \leq |x - y|$ for any real x, y. Then explain that there exist nonzero integers m, n such that $|n - \pi m| < 10^{-9}$.

2.30. We have to show that the distance from some lattice point (a, b) to the line will be less than ϵ. Using the formula for the distance from a point to a line, this is equivalent to proving the existence of integers a, b such that

$$\frac{|ma - b|}{\sqrt{m^2+1}} < \epsilon \quad \Leftrightarrow \quad |ma - b| < \epsilon', \quad \text{where } \epsilon' = \epsilon\sqrt{m^2+1}.$$

2.31. Easy.

2.32. Hint: the statement is equivalent to the following: for each positive integer N exactly one member of $A \cup B$ is in the interval $(N, N+1)$. A great proof for the latter statement is obtained by counting how many members of $A \cup B$ are less than N.

2.33. See Niven [15].

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