Proposed Problem: Large Values of Tan $n$

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1. Proposed Problem

(a). Show that there are infinitely many positive integers $n$ such that

$$|\tan n| > n.$$  \hspace{1cm} (1)

(b). Show that there are infinitely many positive integers $n$ such that

$$\tan n > \frac{1}{4}n.$$  \hspace{1cm} (2)

2. Solution

The proofs require showing that there are integers $n$ sufficiently close to $(2k + 1)\frac{\pi}{2}$ for some integer $k$. We use the following Diophantine approximation lemmas.

(a). \textbf{Lemma 1.} Every irrational real $\theta$ has infinitely many Diophantine approximations $q \equiv 1 \pmod{2}$ with

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{2q^2}.$$  \hspace{1cm} (3)
(b). Lemma 2. Every irrational real $\theta$ has infinitely many (one-sided) Diophantine approximations $q \equiv 1 \pmod{2}$ with

$$\frac{2}{q^2} > \theta - \frac{p}{q} > 0 \ .$$

(4)

Lemma 1 is sharp in the sense that the constant 2 cannot be improved. Similarly Lemma 2 is sharp in that the constant 2 cannot be improved.

Solution. If $n = (2k + 1)\frac{\pi}{2} + y$, where $k$ is an integer and $y$ is small, say $|y| < \frac{1}{4}$, then

$$\tan n = \cot y = 1 + \frac{y^2}{2!} + O(y^4) \ \ \ \$$

$$y - \frac{y^3}{3!} + O(y^5) \ \ \ \$$

$$= \frac{1}{y} + y + O(y^3) \ .$$

(5)

Suppose now that

$$|y| = \left| n - (2k + 1)\frac{\pi}{2} \right| < \frac{1}{\alpha(2k + 1)}$$

(6)

for some constant $\alpha$, then (5) gives

$$|\tan n| > \alpha(2k + 1) - O(1)$$

$$> \frac{2\alpha}{\pi} n - O(1) .$$

To get infinitely many approximations $n$ of the desired quality, we need infinitely many solutions to (6) for fixed $\alpha$ such that:

- **case (a).** $\alpha > \frac{\pi}{2} \approx 1.571$.
- **case (b).** $\alpha > \frac{\pi}{8} \approx .393$, and $n$ satisfies the one-sided approximation condition

$$-\frac{1}{\alpha(2k + 1)} < n - (2k + 1)\frac{\pi}{2} < 0 .$$

(7)

The inequalities (6) and (7) can be rewritten as Diophantine approximations to the irrational $\theta = \frac{\pi}{2}$, namely

$$\left| \frac{\pi}{2} - \frac{n}{2k + 1} \right| < \frac{1}{\alpha(2k + 1)^2} ,$$

(8)

and

$$\frac{1}{\alpha(2k + 1)^2} > \frac{\pi}{2} - \frac{n}{2k + 1} > 0 .$$

(9)
These two cases (a) and (b) are thus covered by Lemmas 1 and 2, respectively. □

**Proof of Lemma 1.** One out of every two convergents \( \frac{p_n}{q_n} \) of the continued fraction expansion of \( \theta = [a_0, a_1, a_2, \ldots] \) satisfies

\[
|\theta - \frac{p_n}{q_n}| < \frac{1}{2q_n^2} .
\]

(See Hardy and Wright, *The Theory of Numbers*, Theorem 183.) At least one of every two consecutive convergents has an odd denominator, since

\[
\det \begin{vmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{vmatrix} = -1 .
\]

If infinitely many consecutive pairs \((q_n, q_{n+1})\) are both odd, then one of each pair satisfies (10), so we are done.

The remaining case is where there are finitely many such pairs. In this case the \(q_n\)’s alternate odd, even, odd, even, \ldots from some point on. Suppose for convenience that the \(q_{2n}\) are all odd, the \(q_{2n+1}\) are all even, from some point on. Then

\[
q_{2n+1} = a_{2n+1}q_{2n} + q_{2n-1} ,
\]

whence \(a_{2n+1} \equiv 0 \pmod{2}\). Thus \(a_{2n+1} \geq 2\) and \(q_{2n+1} \geq 2q_{2n}\). Now [Hardy and Wright, Theorem 164] gives

\[
\left| \theta - \frac{p_{2n}}{q_{2n}} \right| \leq \frac{1}{q_{2n}q_{2n+1}} ,
\]

and

\[
\left| \theta \frac{p_{2n}}{q_{2n}} \right| \leq \frac{1}{q_{2n}(2q_{2n})} \leq \frac{1}{2q_{2n}^2} ,
\]

as required. A similar argument works if all \(q_{2n+1}\) are even and all \(q_{2n+2}\) odd, from some point on. □

**Proof of Lemma 2.** The continued fraction convergents satisfy

\[
\frac{p_{2n}}{q_{2n}} > \theta > \frac{p_{2n+1}}{q_{2n+1}} ,
\]

so if infinitely many \(q_{2n+1}\) are odd, we are done.

Now suppose all \(q_{2n+1}\) are even from some point on, in which case all \(q_{2n}\) are odd from that point on, using (11). As in Lemma 1, we have \(a_{2n+1} \equiv 0 \pmod{2}\) from that point on. Now set

\[
Q := q_{2n+1} - q_{2n} ,
\]

\[
P := p_{2n+1} - p_{2n} .
\]
Certainly \( Q \) is odd, and (12) gives

\[
\frac{P}{Q} = \frac{p_{2n+1} - p_{2n}}{q_{2n+1} - q_{2n}} < \frac{p_{2n+1}}{q_{2n+1}} < \theta .
\] (13)

Next, we have

\[
|Q\theta - P| = |(q_{2n+1}\theta - p_{2n+1}) - (q_{2n}\theta - p_{2n})| \\
\leq |q_{2n+1}\theta - p_{2n+1}| + |q_{2n}\theta - p_{2n}| \\
\leq \frac{1}{q_{2n+2}} + \frac{1}{q_{2n+1}} \\
\leq \frac{2}{q_{2n+1} - q_{2n}} = \frac{2}{Q} .
\] (14)

Now (13) and (14) give

\[
\frac{2}{Q^2} > \theta - \frac{P}{Q} > 0 ,
\]

as required. \( \square \)

Remarks.

(1). Extremal \( \theta \) for Lemma 1 include \( \theta = [a_0, a_1, \ldots] \) with all \( a_{2n+1} = 2 \) and \( a_{2n} \to \infty \) as \( n \to \infty \).

(2). Extremal \( \theta \) for Lemma 2 include \( \theta = [a_0, a_1, \ldots] \) with all \( a_{2n} = 1 \), with \( a_{2n+1} \equiv 0 \mod 2 \) and \( a_{2n+1} \to \infty \) as \( n \to \infty \). [It requires a proof to show that \( Q = q_{2n+1} - q_{2n} \) are the “best” odd denominator one-sided approximations.]

(3). Presumably for each \( \alpha > 0 \) there exist infinitely many positive \( n \) such that

\[
\tan n > \alpha n .
\]

This would be true if \( \frac{\pi}{2} \) were a “random” real number.

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