Surprises, surprises, surprises.[1]

Felix Lazebnik

In great mathematics there is a very high degree of unexpectedness, ...

– G.H. Hardy.

Perhaps the most surprising thing about mathematics is that it is so surprising.

– E.C. Titchmarsh.

My momma always said, "Life is like a box of chocolates. You never know what you're gonna get."

– Forrest Gump[2]

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In my study of mathematics surprises have always played an important part. Here I want to share these experiences with you. I feel a bit uneasy talking about them. They are personal emotions, and others may not feel the same. It is like an old story about two people standing at the edge of a cliff. One looks around and says: "God... What a beauty!". Another looks, and looks, and looks, and then asks: "Where?". The first does not know what to say or do... He just pushes the second off the cliff. Gently.

So, do not stand too close to me...

My surprises have various origins. Here are some examples.

A. "How could I not know this for so long?!"

A good thing about these ones is that you can blame them on your teachers.

B. " Surprising, but not that much, if we look closer... Maybe, I could discover this myself ... "

These are the most numerous for me. Their frequency depends on how cool one is. It depends on how patient one is in "getting to the bottom of this". It depends on one's knowledge of related subjects. It depends on one's self-confidence.

C. "Even after I see a proof, the fact is still a mystery. I could not discover this... "

This depends on how much one likes having mystery in one's life. It depends on one's self-confidence. It depends on one being honest with oneself.

D. "How is it possible that this is not resolved yet?"

Listing some of these, I tried to avoid famous unsolved problems, and those on which I never thought myself. I also picked ones which could be understood by many.

[1]Extended notes of the Lecture at the Graduate Student Seminar. University of Delaware. April 18, 2007. The author is grateful to Jessica Belden for her careful reading of these notes and useful comments.

[2]From the movie *Forrest Gump*, 1994, which won 6 Oscars. Another 32 wins and 37 nominations.

Examples of A: "How could I not know this for so long?"

- **All parabolas are similar.**

Comments.

A figure A in a plane is called **similar** to figure B in the plane, if there exists a positive number k and a bijection $f : A \to B$ such that $\text{dist}(f(x), f(y)) = k \cdot \text{dist}(x, y)$ for any $x, y \in A$. We say that two figures are of the **same shape**, if they are similar. Every two segments are similar, every two circles are similar, every two equilateral triangles are similar. Not every two ellipses are similar, not every two hyperbolas are similar.

So how come all parabolas are similar? Is not it true that $y = x^2$ is "wider" than $y = 10x^2$ and narrower than $y = .01x^2$? NO. The second is just smaller, and the third is larger, but all three are of the same shape. Though stretching or shrinking along y-axis alone does not transform a curve to a similar one, it does for parabolas.

The property is obvious from the definition of a parabola as a locus. The coefficient k is just the ratio of two distances from foci to the corresponding directrix. If one knows that any parabola can be represented in a certain Cartesian coordinate system as a graph of $y = ax^2$, then one can also check this property by considering the transformation $f : (x, y) \mapsto (x', y') = (kx, ky)$. Observe that for $k = a/b$, $y = ax^2 \Leftrightarrow y' = b(x')^2$. So f maps the parabola $y = ax^2$ to parabola $y' = b(x')^2$. Obviously, f is a similarity transformation.

All this appeared and was clarified in our conversations with Owen Byer and Deirdre Smeltzer in January 2007. A few weeks later I stumbled on a translation from Johannes Kepler's "*The foundations of modern optics: Paralipomena to Vitellius, 1604,*" where he mentions this property of parabolas with great excitement!

- **Suppose we have perfectly spherical earth, density is distributed spherically symmetrically, and a cannonball is moving without drag under the influence of the gravitational field. Then the trajectory is a conic section. What is it?**

It is ... ELLIPSE, not parabola, as we are often taught.

Comments.

I learned this recently from an article by Lior M. Burko and Richard H. Price in "*Am. J. Phys.* **73***, June 2005.*", published by the American Association of Physics Teachers.

It immediately made perfect sense to me. I have known for a long time that parabolic trajectory of a projectile is not stable, but, for a strange reason, every thrown stone moved along a parabola... This also made a perfect sense with the fact that that the shape of a very narrow ellipse is close to a parabola: the latter can be considered as an ellipse with one focus removed to infinity. It was also surprising to see in the article a quotation from Newton's "*Principia*", where he points out that Galileo's model, which assumes flat-earth-uniform gravitation, leads to a parabolic trajectory. But the central force model, which is used in astronomy, leads to an ellipse. I do not remember seeing this discussion in the calculus texts ...

Actually, reading about all of this I stumbled at another surprise. I always thought that it is much easier to send a rocket to the Sun, than make it a satellite of Earth, or of Sun, or make it to leave solar system. Not true. It is harder. The escape velocity should be greater than in all latter cases.

One more surprise of this type. I always thought that because of the friction with Earth atmosphere, a satellite slows down. Not true. While its hight decreases, it is moving faster.

- **For a fixed position of Earth and Moon, where do we have high tide, and where do we have low tide?**

Comments.

Answer: Let the line passing through the centers of Earth and Moon intersect Earth in points A and B. Then the high tide is close to these two points, and the low tide is close to points C and D, where segment CD is the diameter perpendicular to the diameter AB.

For some reason, I always thought that if A is farther than B from the center of the Moon, the tide at A is the lowest. This all can be proved by simple computations.

- **When we apply the method of mathematical induction, we use a deductive reasoning.**

Comments.

The deductive method is a passage from general statement to a particular. For example, applying the Pythagorean Theorem to a right triangle with legs 5 and 12, we get (deduce) that the hypothenuse is of length 13. Therefore application of any theorem (including the Mathematical Induction Theorem (Principle)) to a specific statement of the form $\forall n \geq n_0 \; (P(n))$ is a deduction.

- **Characterize the set of all functions f which have continuous nth derivative on an open interval $I \subset \mathbb{R}$ and satisfy the differential equation**

$$f^{(n)} + p_1 f^{(n-1)} + \dots p_{n-1} f' + p_n f = 0,$$

for some continuous functions $p_1, \dots, p_n$ on I.

Comments.

I asked this question when I taught an undergraduate course on Differential Equations. There was an exercise in the book which asked to show that $f(x) = \sin(x^2)$ cannot be a solution of such an equation for $n = 2$. This can be easily seen from the theorem about the uniqueness of the solution of the initial value problem in this case. Indeed, the function which is identically zero on the interval is, obviously, a solution of this equation. As $f(0) = f'(0) = 0$, the function f would represent another solution with the same initial values. Contradiction.

Asking a question above was a natural thing for a person with experience in algebra, but it was surprising to me that this question was not known to people who work with differential equations. And that it was not interesting to them.

I found the answer, but had difficulty proving it. It was David Bellamy who provided the first proof. The question appeared as a Problem # E10729 : On solutions of a class of differential equations, *The American Mathematical Monthly*, Vol. 106, No. 4, 1999. Several people submitted simple solutions, see *The American Mathematical Monthly*, Vol. 107, No. 4 (Apr., 2000), p. 377. For the original (and harder) solution, see

`http://www.math.udel.edu/~lazebnik/papers/DE.pdf` .

- **Find all solutions of $Ax = b$, where $A \in M_{m\times n}(\mathbb{Z})$, $x \in \mathbb{Z}^n$, and $b \in \mathbb{Z}^m$?**

Comments.

The question appeared when I taught a graduate topic course on the Asymptotic Design Theory. It was very surprising to me that having background in algebra, and after teaching Linear Algebra many times, and knowing about the "elementary divisors" of a matrix, I had no idea of how to answer this question. It turned out that my experience was not so unique: the related paper got accepted quickly in the *Mathematic Magazine*. It turned out that the question led H.J.S. Smith to his normal form, and many other people to their famous results. For details, see F. Lazebnik, On Systems of Linear Diophantine Equations, *The Mathematics Magazine*, vol. 69, no. 4, October 1996, 261–266.

Or see `http://www.math.udel.edu/~lazebnik/papers/dior1.pdf`.

- **Find all real values of a such that the sequence $\{a_n\}_{n\geq 0}$ defined by $a_0 = a$, and $a_{n+1} = a_n^2 - 2$ for $n \geq 0$, converges.**

Comments.

I can think of at least four surprises related to this problem.

The first surprising thing about this sequence was its strange behavior, which is completely defined by a. It turns out that if it converges, then it must stabilize: all terms, begining with an arbitrary one, must be equal to its limit, which, obviously, can take one of two values -1 or 2. This allows one to find a. I had not seen anything like this before I asked the question. As I understood later, this was my first exposure to an interesting dynamical system and "repulsors".

The second surprise was the unusual history of the question. I assigned it as one of many other homework problems on limits in a high school where I was working at the time (1977). I had no idea that it was a challenge. I just thought that it would be a nice addition to the standard question: what is the limit of $\{a_n\}_{n\geq 0}$ if it converges? After my students could not do it, I tried it for two days, with no success. There was something special about the sequence... I began asking my colleagues, and, soon, Yuri I. Pilipenko saw the light, and we finished a proof quickly. Several years later, I submitted it as a problem to *Monthly*, and it was accepted: (see Problem # E3036, *The American Mathematical Monthly*, Vol 91, No. 2, 1984).

The third surprise was when I saw solutions and extensive comments sent by *Monthly*'s readers (see *The American Mathematical Monthly*, Vol. 94, No. 8 (Oct., 1987), pp. 789-793). 82 people from 23 countries! It turned out that this type of sequence had been studied long ago, since 1918 at least, and by many mathematicians. The corresponding area (which used to be just analysis) is now called topological dynamics. Many references and generalizations were mentioned. Some people point out that it was a good example of how one can get a wrong answer by using computer.

My fourth surprise: how could it happen that the editors missed the fact that the question had been actually extensively studied? A miracle!

Examples of B: " Surprising, but not that much, if we look closer...
Maybe, we could discover this ... "

- **Watermelon has 99% of water in it. 1 ton of watermelons was shipped, and during the shipment some water evaporated, and in the arrived watermelons water makes 98%. What was the weight of the shipment when it arrived?**

  Comments.

  Well, solve it, as I did. The answer is 0.5 ton. After this problem, my belief of having good intuition about percents was shattered.

- **It takes three days for a motor boat to travel from A to B down the river, and it takes it four days to come back. How long will it take a wooden log to be drifted from A to B by the current?**

  Comments.

  This problem was one among twenty that my mathematics school teacher, Lev Ilyich Bogomolny, assigned for the summer after the eighth grade. I spent a lot of time on it unable to solve it. I was sure that some data were missing. My older brother Lazar helped me with it. I was amazed with the power of algebra. By the way, the answer is 24 days, and it is impossible to find the speed of the current, or the speed of the boat, or the distance AB.

- **Consider any positive integer N whose (decimal) digits read from left to right are in non-decreasing order, but the last two digits (tens and ones) are in increasing order. Prove that the sum of digits of $9N$ is always exactly 9.**

  Comments.

  It was hard to believe, since N could be really large. For example, if $a = 1778$, $b = 2344459$, and $c = 12225557779$, then

  $$9a = 16002,\ 9b = 21100131,\ 9c = 110030020011,$$

  and the sum of digits in each case is 9. The proof is easy. If you find it for for 3- or 4-digit numbers, the generalization is trivial. This problem was communicated to me about three years ago by Valery Kanevsky, a friend and an applied mathematician.

- **Take a 4-digit number with not all digits equal. Rearranging its digits in decreasing order we get a number M. Rearranging its digits in increasing order we get a number m. Consider $M - m$, and repeat the procedure. Do it again, and again... After several iterations we get to the number 6174.**

  Comments.

  If this is not surprising, then I rest my case... I found a proof, but it is not illuminating (case analysis). A similar question can be asked for numbers with arbitrary number of digits, and for bases other than 10. The answer becomes different. Play with computer.

- **$e^{i\pi} + 1 = 0$, or, more general, $e^{ix} = \cos x + i \sin x$.**

Comments.

Please do not become angry with me, as I placed this result in the section "... Maybe, we could discover this". Yes, the equality $e^{i\pi} + 1 = 0$ is considered to be one of the highest standards of mathematical beauty: it ties five the most celebrated and independent mathematical constants: 0, 1, π, e and i!

Well, it clearly follows from $e^{ix} = \cos x + i \sin x$ when $x = \pi$. How mysterious is this? In order to answer this question, one has to assign meaning to powers with complex exponents. How can this be done? Similarly to reals: $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$? We can get the same power series if we define it $e^z := \sum_{n=0}^{\infty} a_n z^n$ and determine unknown coefficients a_n from the "natural" expected properties: $e^{z_1} e^{z_2} = e^{z_1 + z_2}$ and $e^0 = 1$. It is a fun exercise. It is interesting that the formula $e^{ix} = \cos x + i \sin x$ was known before Euler. Roger Cotes published it 34 years earlier. Neither Cotes nor Euler used power series when they arrived to the formula. I do not believe that I could discover it along the paths that they used.

- **100 people board an airplane with 100 seats. Each person has a seat assigned. For some reason, the 1st person who gets in takes her seat at random. Then the 2nd passenger takes her seat if it is not occupied (by the 1st), and picks a seat at random if her seat is occupied. Then the 3rd passenger takes her seat if it is not occupied (by the 1st or 2nd), and picks a seat at random if her seat is occupied. And so on. What is the probability that the last person will sit in her seat?**

Comments.

The answer is 1/2 and it is not hard to prove it. A solution without any computations, can be found in the book by Peter Winkler "*Mathematical Puzzles. A Connoisseur's Collection*", published by A.K. Peters, Ltd. in 2004.

- **There exists a number of the form $111\ldots111$ which is divisible by 2007.**

Comments.

A solution below is an impressive and short application of the Pigeonhole Principle. Here it is. Let $a_1 = 1$, $a_2 = 11$, $a_3 = 111$, and so on, $a_{2007} = 111\ldots111$ (2007 ones). Divide each number by 2007. If one number is divisible (with remainder 0), we are done. If not, two of the remainders must necessarily repeat, as there are at most 2006 distinct nonzero remainders. Subtracting the corresponding numbers, we obtain a number M which is divisible by 2007, and is of the form $111\ldots1111000\ldots000$. Hence $M = N \cdot 10^a$, where N has digits 1 only, and a is the number of zeros in M. Since M is divisible by 2007, and $\gcd(2007, 10^a) = 1$, then 2007 divides N.

It is also surprising that if digit 1 in the desired number is replaced by any sequence of digits (like 348), and 2007 is replaced by any odd integer not divisible by 5, the result will still hold. For example, there exists a number of the form $348348348\ldots348$ which is divisible by 112233.

Examples of C: "Even after I see a proof, the fact is still a mystery."

"I could not discover this...

- **Consider a continuous curve $y = f(x)$ on $[0,1]$ such that $f(0) = f(1) = 0$. A segment joining two points of the curve is called a *chord*. Consider only horizontal chords, i.e., those which are parallel to x-axis. What length can they have?**

  Comments.

  The answer is very striking. It turns out that for any positive integer n, the curve will have a horizontal chord of length $1/n$, and that no other horizontal chord length is guaranteed! The last statement can also be phrased this way: for every $\alpha \neq 1/n$, where n is a positive integer, there exists a curve $y = f(x)$ which satisfies the conditions of the statement and which has no horizontal chord of length α.

  The theorem is attributed to Heinz Hopf. I bet that finding the answer was much harder than proving it. A solution can be found in R.P. Boas, *A Primer of Real Functions*, Fourth Edition, MAA, 1996, or in A.M. Yaglom and I.M. Yaglom, *Challenging mathematical problems with elementary solutions II*, Dover 1967.

- $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

  Comments.

  As we know, this made young Euler a superstar.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \quad \Rightarrow$$
$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n} \quad \Rightarrow$$
$$\frac{\sin \sqrt{x}}{\sqrt{x}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{n}.$$

  Given a polynomial $a_0 + a_1 x + \ldots + a_n x^n$ with nonzero roots $x_1, \ldots, x_n$, we have, by Viète's Theorem,

$$\sum_{i=1}^{n} \frac{1}{x_i} = \frac{\sum_{i=1}^{n} x_1 \cdot \ldots \cdot x_{i-1} x_{i+1} \cdot \ldots \cdot x_n}{x_1 x_2 \cdots x_n} = -\frac{a_1/a_n}{a_0/a_n} = -\frac{a_1}{a_0}.$$

  Looking at $(\sin \sqrt{x})/\sqrt{x}$ as a polynomial of infinite degree (so what?), realizing that its roots are $x_n = \pi^2 n^2$, $n \geq 1$, and applying Viète's Theorem the same way (still holds, of course), we get

$$\sum_{i=1}^{\infty} \frac{1}{\pi^2 n^2} = -\frac{a_1}{a_0} = -\frac{-1/3!}{1} = \frac{1}{6}.$$

  I would never be able to think of this!

- **Let $S(r, R) = \{x \in \mathbb{R}^3 : r \leq \|x\| \leq R\}$ be a uniform density spherical layer. Let A be any point inside it. Then the gravity at A is zero.**

Comments.

If the Law of Gravity had 2 ± 10^{-100} as the exponent in the denominator, this would not be true. Still, Newton had a geometric argument with infinitesimals. It is presented in V.I. Arnold's *"Huygens and Barrow, Newton and Hooke: pioneers in mathematical analysis and catastrophe theory from evolvents to quasicristals"*, Birkhauser Verlag, 1990, but I did not find it convincing. I convinced myself that this was true by using cylindrical coordinates, a triple integral, and Maple. I did not see this problem in Calculus texts. Neither did I see problems asking to demonstrate that solid balls can be replaced by point masses at their centers, when we study motions of planets. I think these are great applications of triple integrals which must be known better.

The question was mentioned to me by Yves Crama, while we were driving on I-295 to UD in 1988. We tried to find a simple explanation for it for several days, but could not.

- **I write two distinct integers, one on a card, and put two cards on the table face down. You can pick any of the two, look at it, and then you have to guess whether the other number is larger or smaller. Prove that you have a strategy to make a correct guess with probability strictly greater than $1/2$.**

Comments.

The first time I heard this question, and its solution, was from Peter Winkler, at a dinner which followed his talk at Penn many years ago. Though the proof was short and convincing, I have difficulties believing the statement. So does every one who I tell this problem to.

For a discussion and a solution, see David Gale's *Tracking the Automatic Ant*, Springer, 1998. He attributes it to David Blackwell's modification of a related question.

- **Alice and Bob have one of two consecutive positive integers n and $n+1$ written on their foreheads. Alice sees Bob's number, and Bob sees Alice's number. Each of them can ask another the question: "Do you know your number?" Suppose Alice and Bob are infinitely intelligent: if there is a way to find out the number on their own forehead, then they will do it. Each of them can answer only "Yes", or "No". Prove that after finitely many question and answers, one of them will know her or his number.**

Comments.

The first time I heard this question, and its solution, was from Peter Winkler, at the same dinner as above. I do not know of a more powerful use of trivial induction.

For a discussion and a solution, see David Gale's *Tracking the Automatic Ant*, Springer, 1998.

- **An** *automorphism* f **of a field is a bijection on it such that** $f(x+y) = f(x) + f(y)$ **and** $f(xy) = f(x)f(y)$**. The only automorphism of** $\mathbb{Q}$ **and** $\mathbb{R}$ **is the identity map. At the same time, there are infinitely many automorphisms of** $\mathbb{C}$**.**

<u>Comments.</u>

I remember this property of $\mathbb{C}$ mentioned by Lev A. Kaluźnin, in one of his lectures on Algebra in early 1970's. It is clear that if we require only continuous automorphisms, then it can be either identity, or conjugation.

The question fascinated me since, but I never seen it discussed in the books, and I never found the time to research it. Finally, I was shown a solution.

If you like this type of question (symmetries of objects), think on two other facts which I found very surprising. For example, the additive groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic, as well as the multiplicative groups $\mathbb{C} \setminus \{0\}$ and $\{z \in \mathbb{C} : |z| = 1\}$.

- **A number** $\alpha \in (0, 1)$ **is called** *normal* **if, for all integers** $b \geq 2$**, when** α **is written in base** b**, each digit appears with the same asymptotic frequency** $1/b$**. Though no explicit example of a normal number is known, the measure of the set of all of them is 1.**

<u>Comments.</u>

A proof by Emil Borel concerning the measure can be found in many analysis and probability texts, or on the web. There are different definitions of normality. For example, one can define a number to be *normal in base* b, if every sequence of k digits appears in it with frequency $1/b^k$, for each positive integer k. For a given b, explicit constructions of normal (in base b) numbers exist. For example,

$$0.12345678910111213\ldots$$

is shown to be normal in base 10. On the other hand, if I am not mistaken, no explicit example of a number normal even in two distinct bases is known. Something is utterly fundamental about the notion of normality. It is related to our intuitive notion of randomness.

Examples of D: "How is it possible that this is not resolved yet?"

- **What is the smallest number of people in the group such that there must be 5 of them who know one another or 5 who do not know one another?**

  Comments.

  The best known result is that this number is between 43 and 49.

  If 5 is replaced by 2, the answer is 2. If 5 is replaced by 3, the answer is 6. If 5 is replaced by 4, the answer is 18.

- **Is it true that if each point in the plane is colored in one of 4 given colors, then there exist two points of the same color and at distance 1 apart?**

  Comments.

  If 4 is replaced by 1, or 2, or 3, the answer is Yes. If 4 is replaced by 7 or a greater number, the answer is No.

- **Are there infinitely or finitely many positive integers n such that**

  $$\tan n > n\ ?$$

  Comments.

  The question was asked by David Bellamy. It is instructive to experiment with Maple, and see that the positive integer solutions are very rare. Together with Jeff Lagarias we could show that each of the inequalities $\tan n < -n$, and $\tan n > \frac{1}{4}n$ have infinitely many solutions in positive integers, but the original problem is still open. See Problem # E10656: On the number of positive integer solutions of $\tan n > n$, *The American Mathematical Monthly*, Vol. 105, No. 4, 1998.

  Or `http://www.math.udel.edu/~lazebnik/papers/tan_n.pdf`

- **How many distinct points of intersection can n lines in a plane have? How many regions can they form?**

  Comments.

  The questions about the *greatest* or the *smallest* number of points (or regions) that n lines in a plane can define are easy. On the other hand, it is not clear which numbers can appear in between.

- **Take an arbitrary invertible $n \times n$ matrix A with entries in $\mathbb{Z}_5$ (field of 5 elements), $n \geq 3$. It is conjectured that there always exists a vector $x = (x_1, x_2, \ldots, x_n)$ with all $x_i \in \mathbb{Z}_5$ such that no x_i is zero, and no component of xA is zero.**

Comments.

The statement is trivial over infinite fields. Noga Alon and Michael Tarsi proved in 1989 that the conjecture is true if $\mathbb{Z}_5$ is replaced by any finite field with nonprime number of elements (more precisely, by $GF(q)$, where $q = p^e \geq 4$, p is prime and $e > 1$). For prime $q \geq 5$, and n much larger than q, the conjecture is still open.