AN INFINITE SERIES OF REGULAR EDGE- BUT NOT VERTEX- TRANSITIVE GRAPHS

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ABSTRACT. Let n be an integer and q be a prime power. Then for any $3 \leq n \leq q-1$, or $n = 2$ and q odd, we construct a connected q-regular edge- but not vertex- transitive graph of order $2q^{n+1}$. This graph is defined via a system of equations over the finite field of q elements. For $n = 2$ and $q = 3$, our graph is isomorphic to the Gray graph.

1. INTRODUCTION

In the following, all graphs are assumed to be simple, i.e. undirected graphs which contain no loops or multiple edges. We say a graph is vertex-transitive if its automorphism group acts transitively on the vertices. Similarly, a graph is edge-transitive if its automorphism group acts transitively on the edges. Edge- but not vertex- transitive graphs are not hard to construct; a simple example would be the complete bipartite graph $K_{m,n}$, with $m \neq n$. Constructing *regular* edge- but not vertex- transitive graphs is much more difficult; they are often referred to as *semisymmetric* graphs. Components of semisymmetric graphs are isomorphic semisymmetric graphs, so the connected case is our main interest. It is known (see [7]) that a semisymmetric graph must be bipartite with equal partition sizes; furthermore, its automorphism group acts transitively on each partition set.

The first constructions of semisymmetric graphs were given by Folkman [7], who initiated the study of such graphs. The list of open questions at the end of this paper stimulated the related research done by others in the many years to come. Most of these questions have been answered, but new and related questions were asked and the investigation continues (see [1] – [6], [8] – [10], [14] – [19]). Though there are some general group-theoretic approaches for construction of semisymmetric graphs, explicit examples of infinite series are rare.

The main purpose of this paper is to provide a new infinite two-parameter family of connected semisymmetric graphs.

2. CONSTRUCTION AND RESULTS

Let q be a prime power, and let P_n and L_n be two $(n+1)$-dimensional vector spaces over the finite field of q elements, denoted by $\mathbb{F}_q$. Elements of P_n will be called *points*, and elements of L_n will be called *lines*. For notational purposes, if $a \in \mathbb{F}_q^{n+1}$ (the cartesian product of $n+1$ copies of $\mathbb{F}_q$), then we write $(a) \in P_n$, and $[a] \in L_n$. The bipartite graph $G_n(q)$ is defined as follows: the vertex set is $P_n \cup L_n$ (where P_n and L_n are the partition sets), and we say a point $(p) = (p_1, p_2, \ldots, p_{n+1})$ is adjacent to a line $[l] = [l_1, l_2, \ldots, l_{n+1}]$ iff the following n relations on their coordinates hold:

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$$\begin{aligned} l_2 + p_2 &= p_1 l_1 \\ l_3 + p_3 &= p_1 l_2 \\ &\vdots \\ l_{n+1} + p_{n+1} &= p_1 l_n. \end{aligned} \tag{2.1}$$

The point (line) with all coordinates equal to 0 will be denoted by (0) ([0]). When (p) is adjacent to $[l]$, we write $(p) \sim [l]$. To refer to such an edge, we write $e = (p)[l]$.

Graphs $G_n(q)$ were known previously. For all n and q prime, they are isomorphic to graphs introduced by Wenger [21]. For all n and prime powers q, graphs $G_n(q)$ are isomorphic to graphs introduced (independently of [21]) by Lazebnik and Ustimenko in [11]. In both [21] and [11], these graphs appeared in the context of extremal Túran-type problems, and were described somewhat differently. In each case, an explicit isomorphism with $G_n(q)$ can be easily furnished.

Our main results are collected in the following theorem.

Theorem 1. *Let q be a prime power. Then*
(1) $G_n(q)$ is a q-regular semisymmetric graph of order $2q^{n+1}$ for any $n \geq 3$ and $q \geq 3$, or $n = 2$ and q odd.
(2) $G_1(q)$ is vertex-transitive; $G_2(q)$ is vertex-transitive for even q.
(3) $G_n(q)$ is connected when $1 \leq n \leq q-1$, and disconnected when $n \geq q$, in which case it has q^{n-q+1} components, each isomorphic to $G_{q-1}(q)$.

We mention that the graph $G_2(3)$ is isomorphic to the Gray graph. Its discovery, according to Bouwer [1] (who rediscovered it independently), is due to Marion Gray in 1932. For different descriptions of the Gray graph see a recent paper by Marušič and Pisanski [17]. It was shown by Malnič, Marušič and Wang [14] that the Gray graph is the only cubic semisymmetric graph of order $2p^3$, where $p \geq 3$ is prime. This implies the claimed isomorphism with $G_2(3)$ (we could also easily verify this by using the computer algebra package Magma). We believe that this simple and compact description of the Gray graph is new. Recently the same group of authors showed in [15] that the Gray graph is the smallest cubic semisymmetric graph.

Proof. (1) It is easy to see that $G_n(q)$ is q-regular. Indeed, the incidence condition (2.1) implies that a neighbor of a point (p) (line $[l]$) is defined uniquely by its first coordinate l_1 (p_1). Therefore each vertex has exactly q neighbors, and $G_n(q)$ is a q-regular graph of order $2q^{n+1}$ and size q^{n+2}.

We will now show the graphs are edge-transitive for all n and q (including the case when $n = 1$). This result appeared in [11] (Corollary to Proposition 3.8), and the idea of a proof suggested there utilized some other results from the paper. Here, we present a complete and self-contained proof.

For each $x \in \mathbb{F}_q$ and each $i = 1, 2, \ldots, n+1$, we define the following mappings $\sigma_i(x), \alpha(x)$ of $V(G_n(q))$ to itself:

$$\begin{aligned} \sigma_1(x) : \quad (p) &\mapsto (p)^{\sigma_1(x)} = (p_1, p_2 + p_1 x, p_3, ..., p_{n+1}), \\ [l] &\mapsto [l]^{\sigma_1(x)} = [l_1 + x, l_2, l_3, ..., l_{n+1}]; \end{aligned}$$

$$\sigma_i(x),\ i \geq 2: \quad (p) \mapsto (p)^{\sigma_i(x)} = (p_1, p_2, ..., p_{i-1}, p_i - x, p_{i+1} + p_1 x, p_{i+2}, ..., p_{n+1}),$$
$$[l] \mapsto [l]^{\sigma_i(x)} = [l_1, l_2, ..., l_{i-1}, l_i + x, l_{i+1}, l_{i+2}, ..., l_{n+1}];$$

$$\alpha(x): \quad (p) \mapsto (p)^{\alpha(x)} =$$
$$(p_1 + x, p_2, ..., p_j + \sum_{i=1}^{j-2} \binom{j-2}{i} p_{j-i} x^i, ..., p_{n+1} + \sum_{i=1}^{n-1} \binom{n-1}{i} p_{n+1-i} x^i),$$
$$[l] \mapsto [l]^{\alpha(x)} =$$
$$[l_1, l_2 + l_1 x, ..., l_j + \sum_{i=1}^{j-1} \binom{j-1}{i} l_{j-i} x^i, ..., l_{n+1} + \sum_{i=1}^{n} \binom{n}{i} l_{n+1-i} x^i].$$

It is a routine verification to show that all $\sigma_i(x)$'s and $\alpha(x)$ are automorphisms of $G_n(q)$.

Let $e = (p)[l]$ be an edge of $G_n(q)$. Consider an automorphism $\sigma = \prod_{i=1}^{n+1} \sigma_i(-l_i)$. Then $[l]^\sigma = [0]$, and $e^\sigma = (p')[0]$, where $(p') = (p'_1, ..., p'_{n+1}) = (p)^\sigma$. Then the incidence equations (2.1) imply that $p'_i = 0$ for all $i = 2, \dots, n+1$. So we have $e^\sigma = (p'_1, 0, ..., 0)[0]$, and therefore

$$e^{\sigma\alpha(-p'_1)} = (p')^{\alpha(-p'_1)}[0]^{\alpha(-p'_1)} = (0)[0].$$

This shows that every edge of the graph can be mapped to the edge $(0)[0]$ by an automorphism; thus $G_n(q)$ is edge-transitive for all n and q. Notice that we have also shown that the automorphism group is transitive on both partition sets.

Next we need to show that these graphs (for the values of n and q specified in statement (1)) are not vertex-transitive. To this end, we exhibit two points with exactly q paths of length four (or 4-paths) between them, while showing that there is no such pair of lines. This proves the graph is not vertex-transitive.

Consider the following 4-path in $G_n(q)$:

$$(0) \sim [x, 0, ..., 0] \sim (y, xy, 0, ..., 0) \sim [z, A_1, A_2, ..., A_n] \sim (0, 1, ..., 1)$$

with $x \neq z$ and $y \neq 0$ (since the first coordinate of a neighbor defines it completely). It is easy to see that $A_i = y^i(z - x) = -1$ for all i. Then $y = 1$ and $x = z + 1$. Thus we have a unique 4-path for any choice of z, and so there are exactly q such 4-paths between these two points.

Since the automorphism group is transitive on lines, when we count the number of 4-paths between two lines, one line can be assumed to be $[0]$. We will also need to invoke the fact ([11], Theorem 3.10) that $G_n(q)$ has no cycles of length less than 8 for all $n \geq 2$ and all q. So consider the following 4-path:

$$[0] \sim (x, 0, ..., 0) \sim [y, xy, x^2y, ..., x^n y] \sim (z, B_1, B_2, ..., B_n) \sim [l_1, l_2, ..., l_{n+1}]$$

with $x \neq z$ and $y \neq 0, l_1$. The absence of 4- and 6-cycles implies that the values of x and y are uniquely determined by any given value of z. In what follows we will show that in such a 4-path, with $[l]$ fixed, there are at most $q-1$ choices for z, and hence at most $q-1$ such 4-paths.

To continue, it is easy to see that $B_i = x^{i-1}y(z-x) = zl_i - l_{i+1}$ for all i. Let $n = 2$ and q be odd. Then B_1 and B_2 simplify to

$$\begin{aligned} y(z-x) &= zl_1 - l_2 \\ xy(z-x) &= zl_2 - l_3. \end{aligned}$$

Case 1: $l_1 \neq 0$. Then to have our 4-path we need $z \neq \frac{l_2}{l_1}$, else either $y = 0$ or $z = x$. Thus the number of 4-paths here is at most $q - 1$.

Case 2: $l_1 = 0$. Then we have

$$\begin{aligned} y(z-x) &= -l_2 \\ xy(z-x) &= zl_2 - l_3. \end{aligned}$$

If $l_2 = 0$, then either $y = 0$ or $z = x$. So let $l_2 \neq 0$. Then $x(-l_2) = zl_2 - l_3$, or $x = -z + \frac{l_3}{l_2}$. So we need $z \neq \frac{l_3}{2l_2}$ (q odd!), otherwise we have $z = x$.

This shows that at least one value for z is always forbidden in the case $n = 2$, and we pass to the case $n \geq 3$.

Remember that B_1, B_2, B_3 are given by

$$\begin{aligned} y(z-x) &= zl_1 - l_2 \\ xy(z-x) &= zl_2 - l_3 \\ x^2y(z-x) &= zl_3 - l_4. \end{aligned}$$

As before we may assume that $l_1 = 0$ and $l_2 \neq 0$. Simple manipulation gives

$$-xl_2 = zl_2 - l_3 \tag{2.2}$$

$$-x^2l_2 = zl_3 - l_4. \tag{2.3}$$

Solving for x in (2.2) and substituting into (2.3) we get

$$-z^2l_2 + 2zl_3 - \frac{l_3^2}{l_2} = zl_3 - l_4.$$

Since $l_2 \neq 0$, the obtained equation is quadratic with respect to z, and so it has at most two solutions. Therefore there are at most two 4-paths in this case. Since $q \geq 3$, the proof of (1) is finished.

(2) From the proof of part (1) of Theorem 1 we saw that both $G_1(q)$ and $G_2(q)$ are point- and line-transitive. Therefore we need only find automorphisms that switch the partition sets. For $G_1(q)$ such an automorphism is given by

$$(p_1, p_2) \mapsto [p_1, p_2], \quad [l_1, l_2] \mapsto (l_1, l_2)$$

and for $G_2(q)$, q even, by

$$(p_1, p_2, p_3) \mapsto [p_1^2, p_3 + p_1p_2, p_2^2], \quad [l_1, l_2, l_3] \mapsto (l_1, l_3, l_2^2 + l_1l_3).$$

Verifications that the maps are automorphisms are immediate in both cases. This proves part (2).

(3) Consider an arbitrary walk from (0): $(0) \sim [v_1, \ldots] \sim (u_1, \ldots) \sim [v_2, \ldots] \sim (u_2, \ldots) \sim \ldots \sim [v_m, \ldots]$, where v_i, u_i are the first coordinates of lines and points

of the walk, respectively. The length of this walk, i.e., the number of its edges, is $2m-1$. We claim that the line $[v_m,\ldots]$ can be represented in the form

$$[v_m, \sum_{i=1}^{m-1} u_i w_i, \sum_{i=1}^{m-1} u_i^2 w_i \ , \ \ldots, \sum_{i=1}^{m-1} u_i^n w_i], \tag{2.4}$$

where $w_i = v_{i+1} - v_i$, $1 \le i \le m-1$.

To see this, we will use induction on m. When $m = 1$ the claim is trivial. Suppose the claim is true for all such walks of odd length less than $2m+1$ ($m \ge 1$). Consider an arbitrary walk from (0) with $2m+1$ edges with labels as above: $(0) \sim \ldots \sim [v_m,\ldots] \sim (u_m,\ldots) \sim [v_{m+1},\ldots]$. Using the inductive hypothesis for the form of $[v_m,\ldots]$ and the adjacency relations (2.1), the point $(u_m,\ldots)$ can be represented as

$$(u_m, u_m v_m - \sum_{i=1}^{m-1} u_i w_i,\ u_m \sum_{i=1}^{m-1} u_i w_i - \sum_{i=1}^{m-1} u_i^2 w_i, \ldots, u_m \sum_{i=1}^{m-1} u_i^{n-1} w_i - \sum_{i=1}^{m-1} u_i^n w_i),$$

and hence the line $[v_{m+1},\ldots]$ can be written in the form

$$[v_{m+1},\ u_m w_m + \sum_{i=1}^{m-1} u_i w_i,\ u_m^2 w_m + \sum_{i=1}^{m-1} u_i^2 w_i, \ldots, u_m^n w_m + \sum_{i=1}^{m-1} u_i^n w_i] =$$

$$[v_{m+1}, \sum_{i=1}^{m} u_i w_i, \sum_{i=1}^{m} u_i^2 w_i \ , \ \ldots, \sum_{i=1}^{m} u_i^n w_i].$$

The claim is proved.

Let C be the component of $G_n(q)$ containing (0). As an immediate corollary of (2.4), we note that the coordinates of any line $[l]$ in C can be described in the following way. Let $n = x(q-1) + y$, where $0 \le y < q-1$. Since $a^q = a$ for any $a \in \mathbb{F}_q$, for $j \ge 2$ the jth coordinate of $[l]$ equals its $(q+j-1)$-st coordinate, i.e.,

$$[l] = [\alpha, a_1, a_2, ..., a_{q-1}, a_1, ..., a_{q-1}, ..., a_1, ..., a_{q-1}, a_1, ..., a_y], \tag{2.5}$$

where we have x groupings of sequences $a_1, ..., a_{q-1}$ in succession. So all lines of C must have the above form, and the first q coordinates of any such line determine it uniquely.

By using (2.5) and the adjacency relations (2.1), one can show that all points in C have a similar form, in that a point is defined by its first q coordinates uniquely.

This also implies that for $n \ge q$, $G_n(q)$ is disconnected. Indeed, consider a line having its second and $(q+1)$-st coordinates distinct. Since there is no walk in the graph between (0) and this line, the graph is disconnected.

Let us now show that for $1 \le n \le q-1$, $G_n(q)$ is connected. Let $[a] = [\alpha, a_1, ..., a_n]$ be a line in $G_n(q)$. To show that there is a walk between (0) and $[a]$, we produce two sequences $v_1, ..., v_{n+1}$ and $u_1, ..., u_n$ of elements of $\mathbb{F}_q$ such that the line given by (2.4) is equal to $[a]$. Letting $w_i = v_{i+1} - v_i$, we see that these u_i's and w_i's will have to satisfy the system:

$$\begin{aligned}
u_1 w_1 + u_2 w_2 + ... + u_n w_n &= a_1 \\
u_1^2 w_1 + u_2^2 w_2 + ... + u_n^2 w_n &= a_2 \\
&\vdots \\
u_1^n w_1 + u_2^n w_2 + ... + u_n^n w_n &= a_n
\end{aligned}$$

Next we assign n distinct non-zero values to the u_i's (this can be done since $n < q$) and try to solve the system with respect to the w_i's.

For $1 \leq n \leq q-2$, the matrix of the system is close to Vandermonde, and it is easy to see that its determinant is $u_1 u_2 \cdots u_n \prod_{i>j}(u_i - u_j)$. For $n = q-1$, the matrix of the system is Vandermonde with rows permuted (the line of all 1's is at the bottom since the $(q-1)$-st power of any nonzero element of the field is 1). Since all u_i's are distinct and nonzero, the determinant is nonzero in both cases and the system has a unique solution. Let it be satisfied by some w_i's. Since $w_i = v_{i+1} - v_i$ for $i = 1, \ldots, n$, setting $v_{n+1} = \alpha$, we can compute $v_1, \ldots, v_n$, and thereby we obtain a walk with $2n+1$ edges joining (0) with $[a]$. Since every point is adjacent to a line, the graph $G_n(q)$ is connected.

For $n \geq q$, an argument similar to the one above shows that every line of the form (2.5) is in C, and there are no other lines in C. Indeed, if $n \geq q$, the i-th and j-th equations of the system coincide for $i \equiv j \bmod (q-1)$. Therefore all equations starting with the q-th can be discarded, and one can repeat the argument given for the case when $n \leq q-1$. This shows that for $n \geq q$, the number of lines in C is q^q. We have seen that in this case the graph is disconnected. Since it is line-transitive, all its components are isomorphic. Therefore, having q^{n+1} lines, the graph has exactly $q^{n+1}/q^q = q^{n-q+1}$ components, each isomorphic to C. Projecting vertices of C to vectors formed by their first q coordinates, one obtains an isomorphism of C to $G_{q-1}(q)$. The proof of the theorem is finished. □

Remark 1. We have seen that when $1 \leq n \leq q-1$, every line of $G_n(q)$ can be reached from (0) by a walk with at most $2n+1$ edges. Since $G_n(q)$ is point-transitive, we obtain that the diameter of $G_n(q)$, $1 \leq n \leq q-1$, is at most $2n+2$. In fact, $2n+2$ is the exact value of the diameter, as was recently shown by Viglione in [20].

Remark 2. In [13] constructions of graphs defined by more general systems of equations similar to (2.1) were considered. There, in Section 2.4, a special type of subgraph of those graphs was presented (a particular case of such a subgraph first appeared in [12]). In $G_n(q)$ these subgraphs can be described as follows. Let A, B be arbitrary subsets of $\mathbb{F}_q$. Set

$$P_{n,A} = \{(p_1, ..., p_{n+1}) \in P_n \mid p_1 \in A\}$$
$$L_{n,B} = \{[l_1, ..., l_{n+1}] \in L_n \mid l_1 \in B\}$$

and let $G_n(q)[A, B]$ be the subgraph of $G_n(q)$ induced on the set of vertices $P_{n,A} \cup L_{n,B}$. Since we restrict the range of only the first coordinates of vertices of $G_n(q)$, $G_n(q)[A, B]$ can alternately be described as the bipartite graph with bipartition $P_{n,A} \cup L_{n,B}$ and adjacency relations as given in (2.1).

If $A = B$ is a subfield of $\mathbb{F}_q$ with $|A| \geq 3$, the graph $G_n(q)[A, A]$ is still semisymmetric: one can mimic the proof of part (1) of Theorem 1! This leads one to the hope that these subgraphs provide additional new examples of semisymmetric graphs. Unfortunately they do not.

Proposition 1. *Let $\mathbb{F}_{q'}$ be a subfield of $\mathbb{F}_q$. Then for $1 \leq n \leq q'-1$, connected components of $G_n(q)[\mathbb{F}_{q'}, \mathbb{F}_{q'}]$ are isomorphic to $G_n(q')$. In particular, $G_n(q')$ is an induced subgraph of $G_n(q)$.*

Proof. All components of $G_n(q)[\mathbb{F}_{q'}, \mathbb{F}_{q'}]$ are isomorphic, so choose the one containing (0). Since $\mathbb{F}_{q'}$ is a subfield of $\mathbb{F}_q$, by considering paths beginning at (0) one sees that any vertex in this component has all its coordinates in $\mathbb{F}_{q'}$. For $1 \leq n \leq q'-1$, $G_n(q')$ is connected, so it is isomorphic to the component. □

It is unclear to us at the moment whether we can obtain further new examples of semisymmetric graphs when $A = B$ is not a subfield of $\mathbb{F}_q$.

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