

Arithmetic and Geometric Sequences

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This collection of problems¹ is for those who wish to learn about arithmetic and geometric sequences, or to those who wish to improve their understanding of these topics, and practice with related problems. The attempt was made to illustrate some ties between these and other notions of mathematics. The problems are divided (by horizontal lines) into three groups according to difficulty: easier, average, and harder. Of course, the division is very subjective. The collection is aimed to freshmen and sophomores college students, to good high school students, and to their teachers.

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An infinite sequence of numbers $\{a_n\}_{n \geq 1}$ is called an **arithmetic** sequence, if there exists a number d , such that for every $n \geq 1$, $a_{n+1} = a_n + d$:

$$\exists d \forall n \in \mathbb{N} (a_{n+1} = a_n + d).$$

The number d is called the **common difference**, or just the **difference** of the arithmetic sequence.

Examples:

$a_1 = 1, d = 1 :$	$1, 2, 3, 4, 5, \dots;$
$a_1 = 12, d = 8.5 :$	$12, 20.5, 29, 37.5, \dots;$
$a_1 = 5, d = -1 :$	$5, 4, 3, 2, 1, 0, -1, -2, \dots;$
$a_1 = 2, d = 0 :$	$2, 2, 2, 2, \dots$

An infinite sequence of numbers $\{b_n\}_{n \geq 1}$ is called a **geometric** sequence, if there exists a number r , such that for every $n \geq 1$, $b_{n+1} = b_n r$:

$$\exists r \forall n \in \mathbb{N} (b_{n+1} = b_n r).$$

The number r is called the **common ratio**, or just the **ratio** of the geometric sequence.

Examples:

$b_1 = 2, r = 1 :$	$2, 2, 2, 2, \dots;$
$b_1 = 1, r = 2 :$	$1, 2, 4, 8, \dots;$
$b_1 = 3, r = 5 :$	$3, 15, 75, 375, \dots;$
$b_1 = -3, r = -1/2 :$	$-3, 3/2, -3/4, 3/8, \dots$

¹Last revision: December 20, 2008.

Often an arithmetic (a geometric) sequence is called an arithmetic (a geometric) **progression**. The term comes from Latin *progredior* – ‘walk forward’; *progressio* – ‘movement forward’, ‘success’. Problems about progressions go back to *Rhind Papyrus*, c. 1550 BC and Babylonian astronomical tables, c. 2500-2000 BC.

- Let $\{a_n\}_{n \geq 1}$ be an arithmetic sequence with difference d . Prove that
 - $a_n = a_1 + (n - 1)d$, for all $n \geq 2$
 - $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$
 - $S_n = a_1 + a_2 + \dots + a_n = \frac{a_1 + a_n}{2}n = \frac{2a_1 + (n-1)d}{2}n$ for all $n \geq 2$ (and for $n = 1$, if we assume $S_1 = a_1$).
- Let $\{b_n\}_{n \geq 1}$ be a geometric sequence with ratio r . Prove that
 - $b_n = b_1 r^{n-1}$, for all $n \geq 2$
 - $b_1 b_n = b_2 b_{n-1} = b_3 b_{n-2} = \dots$
 - Let $n \geq 2$, and $S_n = b_1 + b_2 + \dots + b_n$. Then

$$S_n = \sum_{i=1}^n b_i = \begin{cases} \frac{b_1 - b_n r}{1 - r} = \frac{b_1(1 - r^n)}{1 - r}, & \text{if } r \neq 1 \\ nb_1, & \text{if } r = 1. \end{cases}$$

The formula also holds for $n = 1$, if we assume $S_1 = b_1$.

- If $|r| < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$, and

$$\sum_{i=1}^{\infty} b_i = \frac{b_1}{1 - r}.$$

- Given that each sum below is the sum of a part of an arithmetic or geometric progression, find (or simplify) each sum.
 - $75 + 71 + 67 + \dots + (-61)$.
 - $75 + 15 + 3 + \dots + \frac{3}{5^7}$.
 - $1 + 2 + 3 + \dots + (n - 1) + n$.
 - $i + (i + 1) + (i + 2) + \dots + j$, where $i, j \in \mathbb{Z}$, $i < j$.
 - $1 + 3 + 5 + \dots + (2n - 1)$, where $n \in \mathbb{N}$.
 - $x^i + x^{i+1} + x^{i+2} + \dots + x^j$, where $i, j \in \mathbb{Z}$, $i < j$, $x \neq 1$.
- Suppose that the sum of the first n terms of an arithmetic sequence is given by the formula $S_n = 4n^2 - 3n$ for every $n \geq 1$. Find three first terms of the arithmetic sequence and its difference.

The content of the following two problems explain the terms ‘arithmetic’ and ‘geometric in the context of sequences.

5. Given two numbers a, b , the number $(a + b)/2$ is called their **arithmetic average**.
- Show that every term of an arithmetic sequence, except the first term and the last term (in case of a finite sequence), is the arithmetic average of the preceding term and the following term.
 - Show that if a sequence has the property above, it must be an arithmetic sequence.
 - Generalize the statement in (a) by proving that the k th term is the arithmetic average of the $(k - i)$ th term and the $(k + i)$ th term for all i such that these terms exist.
6. Given two numbers a, b , $ab \geq 0$, the number \sqrt{ab} is called their **geometric average**.
- Show that every term of a geometric sequence with non-negative terms, except the first term and the last term (in case of a finite sequence), is the geometric average of the preceding term and the following term.
 - Show that if a sequence has the property above, it must be a geometric sequence.
 - Generalize the statement in (a) by proving that the k th term is the geometric average of the $(k - i)$ th term and the $(k + i)$ th term for all i such that these terms exist.
7. In a tournament with $n \geq 2$ players every person meets with every other person exactly once. Prove that the number of games played is $n(n - 1)/2$.
8. Prove that if numbers a^2, b^2, c^2 are distinct and form an arithmetic sequence, then the numbers $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ also form an arithmetic sequence.
9. Simplify the ratio of $a + a^2 + \dots + a^{100}$ and $a^{-1} + a^{-2} + \dots + a^{-100}$.
10. Use the formula for the sum $1 + x + x^2 + \dots + x^{m-1}$ to establish the following factorizations of polynomials: for all integer $n \geq 2$, and all integer $k \geq 1$,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1})$$

and

$$a^{2k+1} + b^{2k+1} = (a + b)(a^{2k} - a^{2k-1}b + a^{2k-2}b^2 - \dots - ab^{2k-1} + b^{2k}).$$

Write these formulæ (with all terms present) for $n = 2, 3, 4, 5$, and $k = 1, 2, 3$. As you see, the formulæ generalize the well-known ones: $a^2 - b^2 = (a - b)(a + b)$, and $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

11. Is number $a = 1 + \sqrt[5]{2} + \sqrt[5]{4} + \sqrt[5]{8} + \sqrt[5]{16}$ rational or irrational?

12. Represent the following repeating decimal as the ratio of two integers.

For example,

$$0.3333333 \dots 0.(3) = 1/3,$$

$$0.14285714285714285714 \dots = 0.(142857) = 1/7,$$

$$0.285714285714285714 \dots = 0.(285714) = 2/7,$$

$$0.090909090909090909 \dots = 0.(09) = 1/11,$$

$$0.076923076923076923076 \dots = 0.(076923) = 1/13,$$

$$0.058823529411764705882 \dots = 0.(0588235294117647) = 1/17,$$

(a) $1.2525252525 \dots$

(b) $0.127791791791 \dots$

(c) $4.031313131 \dots$

Comment. From the examples above, one can see that, when p is prime, the length of the principal period in the decimal expansion of $1/p$ is $p - 1$ ($p = 7, 17$), or smaller ($p = 3, 11, 13$). It is an **unsolved problem** whether there are infinitely many primes p such that the principal period in the decimal expansion of $1/p$ has exactly $p - 1$ digits.

13. A positive integer n is called a **perfect number**, if it is equal to the sum of all its positive divisors smaller than n . For example, $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$, so 6 and 28 are perfect numbers.

Euclid proved that the formula $2^{k-1}(2^k - 1)$ gives an even perfect number whenever $2^k - 1$ is prime. Prove this result.

Over a millennium after Euclid (c. 325 BC), Ibn al-Haytham (Alhazen) (c. 1000 AD) realized that every even perfect number is of the form $2^{k-1}(2^k - 1)$ where $2^k - 1$ is prime, but he was not able to prove this result. It was not until the 18th century that L. Euler (1707 - 1783) proved that the formula $2^{k-1}(2^k - 1)$, with $2^k - 1$ prime, will yield all even perfect numbers. Primes of the form $2^k - 1$ are called Mersenne primes (in honor of M. Mersenne (1588 - 1648)). Only 46 Mersenne primes are known as of October 2008, so only 46 perfect numbers are known. We do not know whether there are other Mersenne primes and how many of them exist. Nor we know whether there exists an odd perfect number. These are old **unsolved problems**.

14. Find the sum of all numbers of the form $m/3$, m is a positive integer not divisible by 3, such that

- (a) $\frac{m}{3} \in [5, 9]$.

So we are asked to find the sum

$$\frac{16}{3} + \frac{17}{3} + \frac{19}{3} + \frac{20}{3} + \frac{22}{3} + \cdots + \frac{26}{3}.$$

- (b) $\frac{m}{3} \in [a, b]$, where $a, b \in \mathbb{N}$ and $a < b$. Simplify your answer as much as you can. Check that for $a = 5$, $b = 9$, the result coincides with the one of part (a).
- (c) Let p be a fixed positive prime. Generalize the result of part (b) to the sum of all fractions $m/p \in [a, b]$, where $m, a, b \in \mathbb{Z}$, and $a < b$. Simplify your answer as much as you can. Check that for $p = 3$, and $a, b \in \mathbb{N}$, the result coincides with the one of part (b).
15. Consider a tournament which begins with $n \geq 1$ teams. In the first round, all teams are divided into pairs if n is even, and the winner in each pair passes to the next round (no ties). If n is odd, then one random (lucky) team passes to the next round without playing. The second round proceeds similarly. At the end, only one team is left – the winner.

Examples:

If $n = 7$, then $1 + 6/2 = 4$ teams go to the second round, $4/2 = 2$ teams go to the third round, $2/2 = 1$ team is the winner. The total number of games played is $3 + 2 + 1 = 6$.

Check that if $n = 2^k$, the number of games is $2^k - 1$.

Find a (simple!) formula for the total number of games played in the tournament (and prove it, of course).

16. Consider an n by n square table, with the first row

$$\begin{array}{ccccccc} & 1 & & 2 & & 3 & & \dots & n \\ n + 1 & & n + 2 & & n + 3 & & \dots & 2n \\ 2n + 1 & & 2n + 2 & & 2n + 3 & & \dots & 3n \\ \dots & & \dots & & \dots & & \dots & \dots \\ (n - 1)n + 1 & & (n - 1)n + 2 & & (n - 1)n + 3 & & \dots & n^2 \end{array}$$

Pick a number a_1 from this table, and delete the row and the column it stands in. From the remaining $(n - 1)$ by $(n - 1)$ table, pick a number a_2 , and delete the row and the column it stands. Continue until only one number, a_n , is left.

Show that the sum $a_1 + a_2 + \cdots + a_n$ is always the same and find a simple expression for it.

17. Find the sum and simplify your answer:

$$S_n = 1 \cdot x + 2 \cdot x^2 + \cdots + n \cdot x^n.$$

18. (a) Find the sum

$$\frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \cdots + \frac{1}{103 \cdot 107}.$$

- (b) Numbers a_1, a_2, \dots, a_n , all nonzero, form an arithmetic sequence. Prove that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}.$$

- (c) If the equality above holds for all $n \geq 2$, does it imply that $\{a_n\}_{n \geq 1}$ is an arithmetic sequence?

19. Find a simple expression the sum of all entries of the table

1	2	3	...	k
2	3	4	...	$k+1$
3	4	5	...	$k+2$
...
k	$k+1$	$k+2$...	$2k-1$

20. Find a simple expression for the sum

$$1 - 4 + 9 - 16 + \cdots + (-1)^{n-1} n^2.$$

21. One wants to chose numbers from the infinite sequence

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$$

which form an infinite geometric sequence with the sum equal

$$(i) \frac{1}{5} \qquad (ii) \frac{1}{15}$$

Can this be done? Prove your answer.

22. Three prime numbers greater than 10 form an arithmetic sequence. Prove that the common difference is divisible by 6.

23. Let $a_1, a_2, \dots, a_n, \dots$ be an arithmetic sequence with a nonzero common difference, and let $b_1, b_2, \dots, b_n, \dots$ be a geometric sequence with a positive common ratio. Prove that there exists numbers x and y such that

$$b_n = xy^{a_n} \text{ for all } n \geq 1.$$

(So, the arithmetic and geometric progressions are related after all!).

24. Two trains are 100 miles apart on the same track heading towards each other, one at at 10 mph, and another at 15 mph. At the same time, a SupperBee takes off from the nose of one train at 20 miles per hour, towards the other train. As soon as the bee reaches the other train, it turns around and heads off (at 20 miles per hour) back towards the first train. It continues to do this until the trains collide, killing the bee. What distance does the bee fly before the collision?

25. The following construction leads to the object called the **Koch snowflake**. Begin with an equilateral triangle K_1 having unit side lengths. Then divide each side into three congruent segments, and build an equilateral triangle on the middle segment as the base in the exterior of the original triangle. Then delete the base. We obtain a 12-gon K_2 with every side having length $1/3$. We continue by dividing each side of K_2 into three congruent segments, and building an equilateral triangle on the middle segment as the base in the exterior of K_2 . Then we delete the base, obtaining a polygon K_3 . Applying a similar procedure to K_3 , leads to a polygon K_4 , and so on. See Figure 1 for the first four polygons K_1 , K_2 , K_3 , and K_4 , and Figure 2 – for K_4 enlarged.

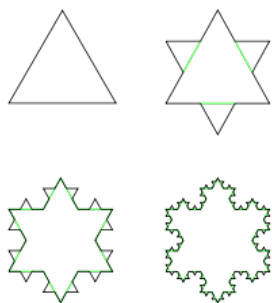


Figure 1: $K_i, i = 1, 2, 3, 4$.

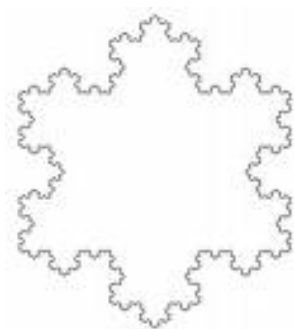


Figure 2: K_4

Compute the following:

- the number of sides of K_n
- the perimeter p_n of K_n
- the area a_n of K_n
- $\lim_{n \rightarrow \infty} p_n$ and $\lim_{n \rightarrow \infty} a_n$.

It can be argued that the sequence of figures K_n “approaches” a certain set of points of the plane, and this set is called the Koch snowflake, in honor of N. F. H. von Koch (1870 - 1924).

26. The construction of the **Sierpinski triangle** begins with an equilateral triangle the unit side length, which we denote by T_1 . The triangle is cut into 4 congruent equilateral triangles, and the central subtriangle is removed. We denote the obtained figure by T_2 . The same procedure is then applied to the remaining 3 subtriangles, leading to a figure T_3 , and so on. See Figure 3.



Figure 3: $T_i, i = 1, \dots, 5$.

It can be argued that the sequence T_n “approaches” a certain set of points of the plane, and this set is called the Sierpinski triangle, in honor of W. Sierpiński (1882 – 1969).

Compute the area of T_n , and show that it tends to zero when n increases.

Comment. The sets of points like Koch snowflake or Sierpinski triangle are examples of objects called **fractals**. Informally a fractal is a figure that can be split into parts, each of which is a reduced-size copy of the whole, a property called **self-similarity**. Fractal-like objects are present in nature. For more on fractals, including beautiful pictures, see

<http://en.wikipedia.org/wiki/Fractal>.

27. Let $x_1, x_2, \dots, x_n, \dots$ be a sequence of distinct numbers from $[0, 1]$. Let B_i denote the interval $[x_i - \frac{1}{2^{i+2}}, x_i + \frac{1}{2^{i+2}}]$, $i \in \mathbb{N}$. Prove that $[0, 1]$ is not a subset of the union of all B_i 's.

Comment. If you are familiar with the notion of cardinality of a set, this statement immediately implies that the set of all points of $[0, 1]$ is uncountable. It provides an alternative to the “diagonal method” for proving this celebrated theorem of G. Cantor (1845 – 1918).

28. Let $d \geq 1$ be the difference in an arithmetic sequence with $n \geq 3$ positive integer terms. Prove that d is divisible by each prime number less than n .

Examples:

3, 5, 7

5, 11, 17, 23

5, 11, 17, 23, 29

7, 37, 67, 97, 127, 157

7, 157, 307, 457, 607, 757

29. Prove that there is no infinite arithmetic sequence with positive difference having all its terms being prime numbers.

Comment. But for every $n \geq 2$, there exist an n -term arithmetic progression consisting of primes only! This remarkable result was proved recently by B. Green and T. Tao, in the paper “The primes contain arbitrarily long arithmetic progressions”, *Annals of Math.* 167 (2008), 481-547.

30. Given n infinite (from both sides) arithmetic sequences with integer terms such that, every two of them share a common term. Prove that all these sequences share a common term.

Hints and Answers.

- (a) and (b) are straightforward. For (c), add S_n with itself (maybe written as $S_n = a_n + a_{n-1} + \dots + 1$), and use (b).
 - (a) and (b) are straightforward. For (c), consider $S_n - rS_n$.
 - Answers: (a) 245 (b) 93.7499904 (c) $(n+1)n/2$
(d) $(i+j)(j-i+1)/2$ (e) n^2 (f) $x^i(1-x^{j-i+1})/(1-x)$
 - Answer: 1, 9, 17. $d = 8$.
 - Straightforward.
 - Straightforward.
 - One can do it in many ways. One way is to reduce the problem to the sum of an arithmetic progression.
 - Straightforward.
 - a^{101}
 - Rewrite $1 + x + x^2 + \dots + x^{m-1} = (1 - x^m)/(1 - x)$, $x \neq 1$, as $x^m - 1 = (x - 1)(x^{m-1} + x^{m-2} + \dots + x + 1)$, substitute $x = a/b$ for $m = n$, or $x = -a/b$ for $m = 2k + 1$, and eliminate the denominators.
 - It is rational. Add the geometric progression.
 - Answers: (a) 124/99 (b) 31916/24975 (c) 3991/990.
 - Hint: If p is prime, consider all distinct divisors of $2^a p$.
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- Answers: (a) 56 (b) $b^2 - a^2$ (c) $(p-1)(b^2 - a^2)/2$.
 - Answer: $n - 1$. Hint: For an arbitrary $n \in \mathbb{N}$, it is easier to solve the problems without reducing it to a geometric sequence.
 - The element in the i th row and the j th column can be written as $(i-1)n+j$. The sum is $n(n^2 + 1)/2$.
 - Use a method similar to the one which gives a formula for the sum of a geometric progression. Another idea is to relate S_n to a derivative of another sum, viewed as a function of x .
 - (a) Answer: 26/107. Hint: $\frac{1}{3 \cdot 7} = \frac{1}{4}(\frac{1}{3} - \frac{1}{7})$, \dots , $\frac{1}{103 \cdot 107} = \frac{1}{4}(\frac{1}{103} - \frac{1}{107})$.
(b) Generalize the idea of the hint to part (a).
(c) Yes.

19. n^3
20. Consider two cases: n is even, and n is odd. Group the terms in pairs.
21. (i) No. (ii) Yes.
22. Hint: show that d is divisible by both 2 and 3.
23. Hint: begin with numerical examples, and then generalize. x and y are defined completely by considering the first two conditions ($n = 1, 2$). Show that x and y found this way “work” also for all $n \geq 3$.
24. 80 miles
25. Answers: (a) $3 \cdot 4^{n-1}$ sides (b) $3 \cdot (4/3)^{n-1}$
(c) $A_n = [8/5 - (3/5)(4/9)^{n-1}](\sqrt{3}/4)$
(d) $\lim_{n \rightarrow \infty} p_n = \infty$, and $\lim_{n \rightarrow \infty} a_n = 2\sqrt{3}/5$
26. Area of $T_n = (\frac{3}{4})^{n-1}(\frac{\sqrt{3}}{4})$.
27. Note that the length of B_i is $1/2^{i+1}$.
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28. Let $a_1 = a, a_2 = a + d, \dots, a_n = a + (n - 1)d$ be the arithmetic sequence with difference d , all a_i are primes, and let p be a prime number less than n . Prove that p divides d .
29. Use the result of Problem 28.
30. Hint: use induction on n .