INVARIANTS I

ROBERT HOCHBERG AND FELIX LAZEBNIK

Rabbit came round to Pooh and looked at him. "Hallo, are you stuck" he asked. "N-no," said Pooh carelessly. "Just resting and thinking and humming to myself."... "The fact is," said Rabbit, "you're stuck."

-A. A. Milne

1. Introduction

We discuss in this note a general method for solving a wide variety of problems. At this state it is not easy to summarize the idea in a concise statement, and we postpone the general discussion for later.

George Polya once said something to the effect that a trick applied twice becomes a method. We completely share this opinion and invite the reader to follow us through several examples from which the promised approach emerges.

2. Examples from Discrete Mathematics

Example 1. *Integers* $1, 2, \ldots, 9, 10$ *are written on a board. John picks any two of the numbers, deletes them, and writes on the board the absolute value of their difference. He repeats this procedure with the resulting 9 numbers, and so on. After he does it 9 times only one number remains on the board. Can this number be 4?*

Solution 1. No, it cannot. Let us explain why. Let's refer to the procedure John performs as a "move". Originally there are 5 odd numbers on the board. How does this quantity change with each move? If two even numbers are chosen, their difference is even, and the number of odd numbers on the board does not change after the move. If an even and an odd numbers are deleted, then their difference is odd, and the number of odd numbers on the board does not change after the move. If two odd numbers are deleted, then the number of odd numbers on the board decreases by 2 after the move. Each time a move is made, the number of odd numbers on the board either does not change or decreases by two. Since its initial value is 5 – an odd number, it will be odd after every move. If 4 were the last number on the board, then the number of odd numbers on the board at that moment was zero. But zero is an even number. Therefore it is not possible to have 4 as the last number. □

Example 2. *Wendy tore up a sheet of paper into 8 pieces, then tore some of these into 8 pieces, and so on, several more times. Can she obtain exactly 1998 pieces at the end?*

Date: February 11, 2004.

Solution 2. No, it cannot. Let us refer to tearing up a piece of paper into 8 new ones as a "move". The number of pieces thus obtained does not depend on whether Wendy tears up several pieces at once or one at a time, and we assume that she does one move at a time. Each time a move is performed, the total number of pieces is increased by 7 ($-1+8=7$.) So after each move, if we divide the total number of pieces by 7, we should get the same remainder each time. For the original number 8, this remainder is 1. For 1998, this remainder is 3: $285 \times 7 + 3$ which is different from 1. This proves that the total number of pieces will never equal to 1998. □

Example 3. *Is there a telephone network consisting of 15 phones in which each phone is directly wired to exactly 3 other phones?*

Solution 3. No, there is not. Clearly every network can be built by starting with 15 isolated phones (no connections at all) and then adding 1 wire at a time (a "move") between two phones. Let us number the phones $1, 2, \ldots, 15$, and let d_i denote the number of phones to which phone i is directly wired. Finally, let $d = d_1 + d_2 + \cdots + d_{15}$. (Note that d, as well as the d_i's will change as we make moves.) In the initial network, all $d_i = 0$, and so $d = 0$. After a move is performed, exactly two of d_i's increase by 1, and therefore d increases by 2 with every move. Thus the value of d is an even after any number of moves. In the network we are looking for, each $d_i = 3$ and $d = 3 \times 15 = 45$ – an odd number. Therefore it does not exist. □

Before proceeding further, let us analyze the the solutions of Examples 1-3. In each case we interpreted the problem as a sequence of allowable "moves" between elements of a certain set S: the set of "states" or "configurations" associated with the problem. Each time the moves were defined within the solution. What about these various sets S?

In Example 1, the set S can be defined as a set of all collections of non-negative integers (repetitions of elements in a collection are allowed).

In Example 2, S is the set of all positive numbers.

Example 3 differed from the previous two in that no procedure is mentioned in the problem, and it was our task, in modelling the problem, to introduce moves and states. The set of states S is the set of all networks with 15 phones.

The only requirement for S is that it has to contain all states which can be produced from the original one by allowable moves. In Examples 1 and 2 above, our stated set S contains some states which cannot be reached by allowable moves. This is okay.

But one will agree that the modelling of the problems as a sequence of moves between states was not the key feature of our solutions. The most important part was the introduction of a certain numerical function on S with the property that, as one moved from state to state via allowed moves, the function value *did not change*. We call such a function an *invariant*, and denote it by Inv. Thus $Inv(s) = Inv(s')$ for any states $s, s' \in S$, where s' is reachable from s by a move.

In Example 1, Inv can be defined to be the remainder obtained when the number of odd numbers on the board is divided by 2. More explicitly, $Inv(s) = 0$ if the collection s of numbers on the board contains an even number of odd numbers, and $Inv(s) = 1$ otherwise. (The function which gives the remainder when dividing by 2 is often called the *parity* function. Thus we can say that $Inv(s)$ in this example is the parity of the quantity of odd numbers in s.) Another invariant one might choose for this example is *the parity of the sum of the numbers on the board.*

In Example 2, Inv can be defined to be the remainder obtained by dividing the number of pieces of paper by 7.

In Example 3, $Inv(s)$ can be defined to be the parity of d for a network s, where $d = d_1 + d_2 + \cdots + d_{15}$.

In all of the examples above, a negative answer followed from the fact that the value of inv on the initial state was different from the value on the state described in the question. Therefore this state could not be reached via allowable moves.

Let us continue with examples.

Example 4. *On a table there are 7 scrabble letters (A through G) lined up in a neat row, in reverse order, as shown below.*

$$G \quad F \quad E \quad D \quad C \quad B \quad A$$

At each step, you may switch the positions of any two adjacent letters. Can we put this list of 7 letters into alphabetical order in exactly 50 steps?

Solution 4. After a little experimentation one finds that although it is not difficult to place the pieces into alphabetical order, the number of moves we perform is always odd. If we can prove this, then the answer to the question will be negative. Why don't we try?

We model the problem as we did the previous ones, letting the set S of states be the set of permutations of these 7 letters (all $7! = 5040$ of them), and calling an interchange of any two adjacent letters in a permutation an "allowable move." What invariant can we find here?

We will say that a pair of letters from a list forms an *inversion*, if the letters appear in the wrong order (meaning that an alphabetically later letter comes *before* an alphabetically earlier one). To form an inversion, the letters do not have to be adjacent in the list. For example the list BDEACGF has 6 inversions: B and A, D and A , D and C, E and A, E and C, G and F.

For any permutation $s \in S$, let $i(s)$ denote the number of inversions in s. (Recall that i counts *all* pairs which are in the wrong order, not just adjacent pairs.) To start with, $i = 21$ for the given list GFEDCBA. (To count it fast, one can see that it is $\binom{7}{2} = 21$, because every pair of letters occurs in the wrong order.)

Now, i is not an invariant, neither the parity of i. But the careful observer will notice that at each step, i will change by *exactly 1*, either increasing or decreasing. This is the hook that we need! You see, since i is required to change by 1 at each step, that means that i will change parity (odd or even) at each step. i.e., as we take steps i starts odd, and then alternates even, odd, even, odd, even, odd etc... This means that the sum $i + N$ will always be odd, i.e., the parity of this number will always be 1. This is summarized in the table below:

Number of moves, N	even	odd	even	odd	even	odd	even	$\cdots$
Number of inversions, i	odd	even	odd	even	odd	even	odd	$\cdots$
Sum $N + i$	odd	odd	odd	odd	odd	odd	odd	$\cdots$

So $i(s) + N$ is always odd. If it were possible to achieve the alphabetized list in exactly 50 steps, we would have i (ABCDEFG)+50 $= 0 + 50 = 50$, which is even. This means that no one can produce the reversed list in 50 steps. □

As we mentioned (and you convinced yourselves that it is true), if you switch two adjacent items in a permutation, then the number of inversions will change by exactly 1, and so therefore the parity of the permutation will change.

It turns out that a switch of *any two items* in a permutation, adjacent or not, changes its parity. This is because we can "simulate" a switch of non-adjacent elements by several adjacent switches. For suppose two non-adjacent elements x and y have exactly m elements between them in the permutation:

$$\dots x a_1\, a_2 \dots a_m y \dots$$

Let us switch x with a_1, then with a_2, then with a_3, and so on until we switch it with a_m, and then with y. This is a total of $m+1$ adjacent switches so far. Then we move y to the left via adjacent switches with $a_m, a_{m-1}, \dots, a_1$ until it is where x started out. This is an additional m transpositions, and results in a switch of the originally non-adjacent elements x and$y$ in $2m+1$ adjacent switches. But as we showed above, each of these adjacent switches changed the parity; and since $2m+1$ is odd, the parity was changed an odd number of times, resulting in a net parity change for the whole operation.

Example 5. *Our next example is the famous "Fifteen Puzzle" by Sam Loyd. (More on this puzzle can be found in Matters Mathematical [6] by Herstein and Kaplansky.*

1	*2*	*3*	*4*
5	*6*	*7*	*8*
9	*10*	*11*	*12*
13	*15*	*14*	

$\longrightarrow$

1	*2*	*3*	*4*
5	*6*	*7*	*8*
9	*10*	*11*	*12*
13	*14*	*15*	

Here we are given a 4×4 box with fifteen 1×1 square blocks in it, numbered 1 through 15 as shown above, with one empty square into which other blocks may be moved. If one starts with the configuration shown on the left, is it possible to slide the blocks around to achieve the "solved" position shown to the right?

Solution 5. First, instead of dealing with an empty space, we can think of it as a 16th virtual block having number 16 on it. Then every allowable move can be thought of as a switch with this "block 16". Next, instead of working with the 4×4 board, we can imagine that all sixteen blocks are lined up in a 1×16 row formed by placing the 1×4 rows end-to-end. So every board position corresponds to a permutation of these sixteen blocks, and every move on the board corresponds to a switch of a certain element of the permutation with our block 16.

Starting with a permutation

$$a = 123456 \dots 13\ 15\ 14\ 16,$$

one wants to obtain the 'standard' permutation

$$b = 123456 \dots 13\ 14\ 15\ 16,$$

by using allowable moves only.

Now, in order to prove theorems it frequently becomes necessary to glean information from any and every available source, use that information as cleverly as possible, and draw the conclusion required by the problem. Here, we will be looking at this puzzle in both of the forms mentioned above: As a 1×16 permutation and as a 4×4 block puzzle. Let us show that the problem posed is impossible.

Suppose this *can* be done, and consider the number of moves that some alleged solution uses. Looking at this as a 4×4 block puzzle we can conclude that the number of moves must be even. For, since 16 starts and ends its journey around the board in the same position on the board (the lower right corner), it moves left as many times as it moves right, and it moves up as many times as down. Since every move really does move 16 in one of these directions, the total number of moves is even. Now let's consider the corresponding permutations. Each of the moves in the board corresponds to a switch in the permutation, so that our alleged solution must correspond to an even number of switches in the permutation. According to our discussion which preceded this example, each switch changes the parity of the permutation. Hence an even number of switches leads to a permutation of the same parity. Well, the number of the inversions in a is one, and in b is zero, so they are of distinct parity. The obtained contradiction implies that the required transformation is not possible. □

When this is taught in a classroom it takes the students about thirty seconds to come up with the question: "how do we invent an invariant for a given problem?" We do not have an answer. It seems to be an art, as is problem solving itself. One improves on it by following examples and by trying, failing, and sometimes succeeding. We would like to mention that the method is not universal. At the same time, one often stumble son a solution when an invariant is used in the most unexpected setting. So it is a good technique to keep in mind.

More examples and useful discussions can be found in the article by Y. Ionin nd L. Kurlyandchik [4] and in the book by D. Fomin, S. Genkin and I. Itenberg [3].

EXERCISES

(1) There are 7 glasses on a table–all standing upside down. You are allowed to turn over any 4 of them in one move. Is it possible to have all the glasses right-side-up?

(2) The numbers $1, 2, \ldots, 1998$ are written on the board. Two numbers are erased and replaced by the remainder of their sum when divided by 13. This operation is repeated until one number is left. What is the number?

(*Hint*: Consider remainders under division by 13, of the sum of all numbers written on the board.)

(3) You have a 4×4 square grid with -1's in the top row, -1's in the leftmost column, and +1's everywhere else. You can change the sign of all the entries of any row or any column on each step. Show that you can never make this matrix be entirely positive (i.e., make all its entries +1.

-1	-1	-1	-1
-1	+1	+1	+1
-1	+1	+1	+1
-1	+1	+1	+1

(4) Begin with an 8×8 checkerboard and prune it by cutting out two opposite corners. Is it possible to tile the remaining 62-square board with 1×2 dominos? (Each domino covers 2 exactly two adjacent squares, no gaps or overlaps allowed.)

(*Hint:* Assume that the squares of the original checkerboard were colored in the usual pattern, say in white and black. Consider the difference $W-B$, where W and B are the numbers of white and black squares covered by already placed dominos.)

(5) You are given 25 copies of a 1×4 tile, with which to cover all the squares of a 10×10 square grid, without gaps or overlaps. Can this be done?

(6) Recall the Euclidean algorithm – an algorithm of finding the greatest common divisor of two integers. Can you vew it in terms of "states", "allowable moves", "invariant"?

(7) Eleven numbers $-12, -9, -8, -7, 1, 2, 3, 4, 5, 6, 11$ are written on a board. John picks any two of the numbers, deletes them, and writes on the board the absolute value of their difference. He repeats this procedure with the resulting 10 numbers, and so on. After he does it 10 times only one number remains on the board. Can this number be 7?

(8) Wendy can tear up a sheet of paper into 4 or 7 pieces, then tear some of these into 4 or 7 pieces pieces, and so on, several more times. Can she obtain exactly 2003 pieces at the end? Does the answer change if she is allowed to tear a piece into 5 or 10 pieces?

(9) Start with a completely disconnected graph consisting of n vertices and 0 edges. Now start building a graph by adding edges, with the restriction that you do not create any cycles. Let k denote the number of connected components at any stage, and let e denote the number of edges which have so far been placed at any stage. Show that the quantity $c = k + e$ is an invariant. Then prove that any connected graph on n vertices which has no cycles (a tree) must have exactly $n-1$ edges.

(10) You are given 3 rows of coins, laid out as follows:

1	5	10	25
1	10	5	25
25	10	5	1

Starting at the first row, you switch the positions of any adjacent pair of coins that you wish, then you go to the second row, and again you switch the positions of any adjacent pair that you wish, then the third row, switch, then back to the first row, switch, then the second row, switch, and so on for as long as you wish. Your goal is to have each row end up in increasing order, the way the first row is to start with. Can this be done?

References

[1] G. Chartrand, *Introductory Graph Theory*, Dover Publications, Inc., New York, 1977.

[2] R. Bellman, K. L. Cook, J. A. Lockett, *Algorithms, graphs and Computers* Academic Press, New York and London, 1970.

[3] D. Fomin, S. Genkin, I. Itenberg, *Mathematical Circles (Russian Experience)*, Mathematical World, Vol. **7**, American Mathematical Society, 1996.

[4] Y. Ionin and L. Kurlyandchik *Some things never change*, Quantum September/October 1993.

[5] I. N. Herstein, I. Kaplansky *Matters Mathematical*, Second Edition, Chelsea Publ. Company, New York, 1978.

3344 Landmark Street, Unit I-3, Greenville, NC 27834
E-mail address: hochberg@dimacs.rutgers.edu

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
E-mail address: lazebnik@math.udel.edu