Short Proofs of Two Basic Properties of Central Projections

Felix Lazebnik

Department of Mathematical Sciences

University of Delaware, Newark, DE 19716

January 19, 2004

In this note we prove two important properties of central projections, stated as Theorems A and B. They can be applied to obtain simple solutions of many hard problems in Euclidean geometry: for numerous examples see Yaglom [1, Ch. 1.3]. The proofs of these properties that I found in the literature were of two kinds: those which were completely elementary, assumed proficiency with spacial Euclidean geometry, and were not easy, and those where the properties followed from more general results of projective geometry. The latter required substantial background in projective geometry.

In my opinion, the proofs presented below are elementary and much easier than the ones I have seen. They can be presented to students familiar with the basics of the analytical geometry in space, namely with equations of circles and lines.

Theorem A. *Given a circle C in plane π and a point D inside of C, one can always find a point S and a plane π' such that the central projection of π to π' from S maps C onto a circle C' and D into its center D'.*

Theorem B. *Given a circle C in plane π and a line l not intersecting C, one can always find a point S and a plane π' such that the central projection of π to π' from S maps C onto a circle C' and l onto l'_∞ – the line at infinity of π'.*

We start by giving a short proof of Theorem A. A very similar argument can be used to prove Theorem B, and it is presented next.

Proof of Theorem A If D is a center of C, the claim is obvious: take $\pi = \pi'$ and S being any point not in π. If D is not a center of C, we introduce a coordinate system $OXYZ$ such that C is a unit circle in XOY and D has coordinates $(d, 0, 0)$, for some d, $0 < d < 1$. Hence π is the OXY-plane and its equation is $z = 0$.

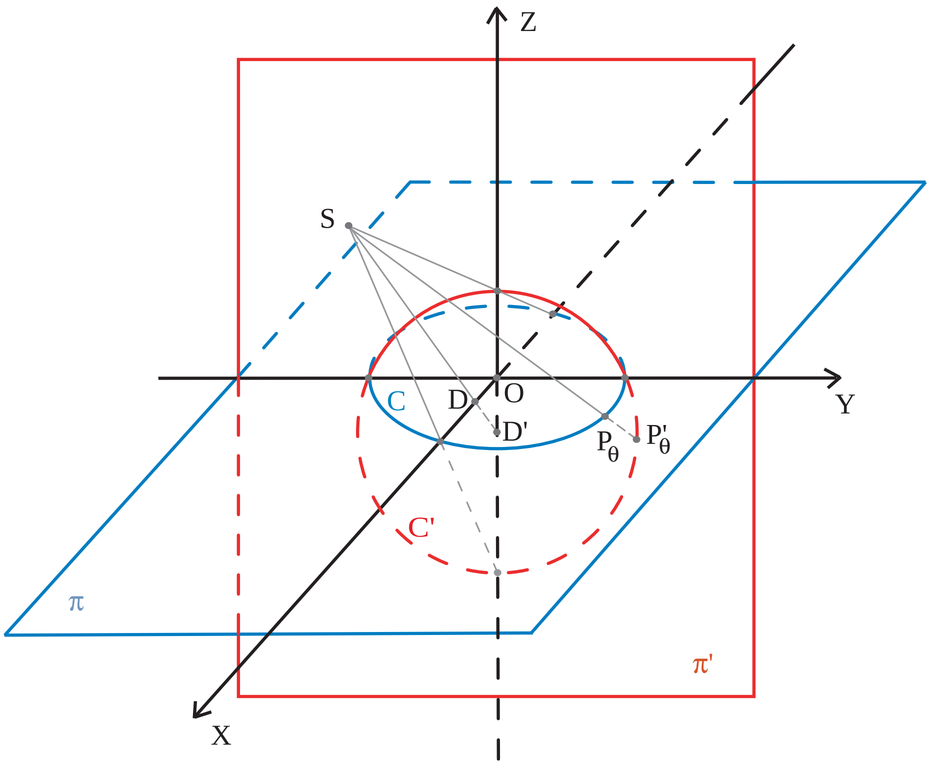

Figure 1: Central projection for Theorem A

Let π' be the OYZ-plane and S be the point with coordinates $(a, 0, b)$, where $a = 1/d$ and $b = \sqrt{1-d^2}/d$. We will show that the central projection of π to π' from S has the required properties.
Indeed, let $P_\theta = (\cos\theta, \sin\theta, 0)$, $0 \leq \theta < 2\pi$, be a parametrization of C. The parametric equation of the line SP_θ is:

$$x = a + (a - \cos\theta)s,\ y = -\sin\theta\, s,\ z = b + bs, \quad s \in \mathbb{R}$$

Plane π' has an equation $x = 0$. Hence point P'_θ – the image of P_θ, and point D' – the image of D, have coordinates

$$P'_\theta \left(0,\ \frac{\sin\theta}{d\cos\theta - 1},\ \frac{\sqrt{1-d^2}\cos\theta}{d\cos\theta - 1}\right)$$

$$D' \left(0,\ 0,\ -\frac{d}{\sqrt{1-d^2}}\right)$$

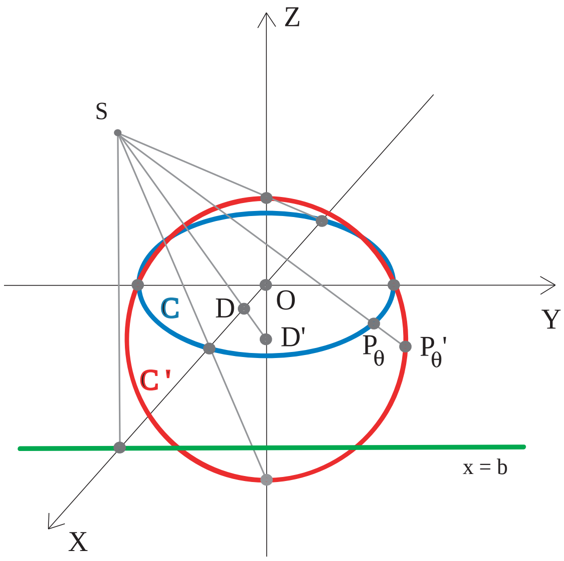

Figure 2: Central projection for Theorem B

Then the square of the distance $D'P'_\theta$ is (check!):

$$D'P'^2_\theta = 0^2 + \left(\frac{\sin\theta}{d\cos\theta - 1} - 0\right)^2 + \left(\frac{\sqrt{1-d^2}\cos\theta}{d\cos\theta - 1} + \frac{d}{\sqrt{1-d^2}}\right)^2 = \frac{1}{1-d^2}.$$

Therefore the distance between D' and P'_θ is $\frac{1}{1-d^2}$ for all θ. Hence C' is a circle (by continuity), and D' is its center. $\square$

Proof of Theorem B We introduce a coordinate system $OXYZ$ such that C becomes a unit circle in the OXY-plane, and l is in OXY-plane having an equation $x = b$, for some b, $b > 1$. Hence π is the OXY-plane and its equation is $z = 0$.
Let π' be the OYZ-plane and S be the point with coordinates $(b, 0, \sqrt{b^2-1})$. As was demonstrated in the proof of Theorem A, set $d = 1/b$, when C is centrally projected from S to π', its image is a circle. Since the plane defined by l and S is parallel to π', l is mapped to the line at infinity of π'. $\square$

The proof of Theorem A leaves an important question unanswered: how did we know how to choose π' and S, i.e., why did we choose $\pi' :\ x = 0$, and $(a, 0, b) = (1/d, 0, \sqrt{1-d^2}/d)$?

The truth is that these values of a and b were originally found as a result of some more tedious computations using computer and Maple. The details can be found in a longer version of this manuscript [3].

To end this note, I would like to mention a fact which sparked my interest in the topic. This was a sketch of D. Hilbert's proof from M. Kac and S.M. Ulam [2] on the impossibility of constructing the center of a circle by using a straight edge only. I invite the reader to find a proof. Having Theorem A makes it an easy exercise.

REFERENCES

1. Yaglom, I.M., Geometric Transformations III, The Mathematical Association of America, New Mathematical Library 24, 1973.

2. Kac M., Ulam S., Mathematics and Logic, Dover Pub. Inc., 1992.

3. Lazebnik, F., `http://www.math.udel.edu/ lazebnik/centralprojection.pdf`