Let $q = p^e$, where $p$ is a prime and $e \geq 1$ is an integer. For $m \geq 1$, let $P$ and $L$ be two copies of the $(m+1)$-dimensional vector spaces over the finite field $\mathbb{F}_q$. Consider the bipartite graph $W_m(q)$ with partite sets $P$ and $L$ defined as follows: a point $(p) = (p_1, p_2, \ldots, p_{m+1}) \in P$ is adjacent to a line $[l] = [l_1, l_2, \ldots, l_{m+1}] \in L$ if and only if the following $m$ equalities hold: $l_{i+1} + p_{i+1} = l_ip_i$ for $i = 1, \ldots, m$. We call the graphs $W_m(q)$ Wenger graphs. In this paper, we determine all distinct eigenvalues of the adjacency matrix of $W_m(q)$ and their multiplicities. We also survey results on Wenger graphs. © 2014 Elsevier Inc. All rights reserved.
\begin{align*}
l_{m+1} + p_{m+1} &= l_mp_1.
\end{align*}

The graph \( W_m(q) \) has \( 2q^{m+1} \) vertices, is \( q \)-regular and has \( q^{m+2} \) edges.

In [25], Wenger introduced a family of \( p \)-regular bipartite graphs \( H_k(p) \) as follows. For every \( k \geq 2 \), and every prime \( p \), the partite sets of \( H_k(p) \) are two copies of integer sequences \( \{0, 1, \ldots, p-1\}^k \), with vertices \( a = (a_0, a_1, \ldots, a_{k-1}) \) and \( b = (b_0, b_1, \ldots, b_{k-1}) \) forming an edge if

\[
b_j \equiv a_j + a_{j+1}b_{k-1} \pmod{p} \text{ for all } j = 0, \ldots, k-2.
\]

In [9], Lazebnik and Ustimenko, using a construction based on a certain Lie algebra, arrived at a family of bipartite graphs \( H'_n(q), n \geq 3, q \) is a prime power, whose partite sets were two copies of \( \mathbb{F}_q^{n-1} \), with vertices \( (p) = (p_2, p_3, \ldots, p_n) \) and \([l] = [l_1, l_3, \ldots, l_n]\) forming an edge if

\[
l_k - p_k = l_1p_{k-1} \text{ for all } k = 3, \ldots, n.
\]

It is easy to see that for all \( k \geq 2 \) and prime \( p \), graphs \( H_k(p) \) and \( H'_{k+1}(p) \) are isomorphic, and the map

\[
\phi : (a_0, a_1, \ldots, a_{k-1}) \mapsto (a_{k-1}, a_{k-2}, \ldots, a_0),
(b_0, b_1, \ldots, b_{k-1}) \mapsto [b_{k-1}, b_{k-2}, \ldots, b_0],
\]

provides an isomorphism from \( H_k(p) \) to \( H'_{k+1}(p) \). Hence, graphs \( H'_n(q) \) can be viewed as generalizations of graphs \( H_k(p) \). It is also easy to show that graphs \( H'_{m+2}(q) \) and \( W_m(q) \) are isomorphic: the function

\[
\psi : (p_2, p_3, \ldots, p_{m+2}) \mapsto [p_2, p_3, \ldots, p_{m+2}],
[l_1, l_3, \ldots, l_{m+2}] \mapsto (-l_1, -l_3, \ldots, -l_{m+1}),
\]

mapping points to lines and lines to points, is an isomorphism of \( H'_{m+2}(q) \) to \( W_m(q) \). Combining this isomorphism with the results in [9], we obtain that the graph \( W_1(q) \)
is isomorphic to an induced subgraph of the point-line incidence graph of the projective plane $PG(2,q)$, the graph $W_2(q)$ is isomorphic to an induced subgraph of the point-line incidence graph of the generalized quadrangle $Q(4,q)$, and $W_3(q)$ is a homomorphic image of an induced subgraph of the point-line incidence graph of the generalized hexagon $H(q)$.

We call the graphs $W_m(q)$ Wenger graphs. The representation of Wenger graphs as $W_m(q)$ graphs first appeared in Lazebnik and Viglione [11]. These authors suggested another useful representation of these graphs, where the right-hand sides of equations are represented as monomials of $p_1$ and $l_1$ only, see [22]. For this, define a bipartite graph $W'_m(q)$ with the same partite sets as $W_m(q)$, where $(p) = (p_1,p_2,\ldots,p_{m+1})$ and $[l] = [l_1,l_2,\ldots,l_{m+1}]$ are adjacent if

$$l_k + p_k = l_1 p_1^{k-1} \quad \text{for all } k = 2,\ldots,m+1.$$  

The map

$$\omega : (p) \mapsto (p_1,p_2,p'_2,\ldots,p'_{m+1}), \quad \text{where } p'_k = p_k + \sum_{i=2}^{k-1} p_i p_1^{k-i}, \quad k = 3,\ldots,m+1,$$

$$[l] \mapsto [l_1,l_2,\ldots,l_{m+1}],$$

defines an isomorphism from $W_m(q)$ and $W'_m(q)$.

It was shown in [9] that the automorphism group of $W_m(q)$ acts transitively on each of $P$ and $L$, and on the set of edges of $W_m(q)$. In other words, the graphs $W_m(q)$ are point-, line-, and edge-transitive. A more detailed study, see [11], also showed that $W_1(q)$ is vertex-transitive for all $q$, and that $W_2(q)$ is vertex-transitive for even $q$. For all $m \geq 3$ and $q \geq 3$, and for $m = 2$ and all odd $q$, the graphs $W_m(q)$ are not vertex-transitive. Another result of [11] is that $W_m(q)$ is connected when $1 \leq m \leq q-1$, and disconnected when $m \geq q$, in which case it has $q^{m-q+1}$ components, each isomorphic to $W_{q-1}(q)$. In [23], Viglione proved that when $1 \leq m \leq q-1$, the diameter of $W_m(q)$ is $2m + 2$.

We wish to note that the statement about the number of components of $W_m(q)$ becomes apparent from the representation (1). Indeed, as $l_1 p_1^i = l_1 p_1^{i+q-1}$, all points and lines in a component have the property that their coordinates $i$ and $j$, where $i \equiv j \mod (q - 1)$, are equal. Hence, points $(p)$, having $p_1 = \cdots = p_q = 0$, and at least one distinct coordinate $p_i$, $q + 1 \leq i \leq m + 1$, belong to different components. This shows that the number of components is at least $q^{m-q+1}$. As $W_{q-1}(q)$ is connected and $W_m(q)$ is edge-transitive, all components are isomorphic to $W_{q-1}(q)$. Hence, there are exactly $q^{m-q+1}$ of them. A result of Mader [16] also obtained independently by Watkins [24], and the edge-transitivity of $W_m(q)$ imply that the vertex connectivity (and consequently the edge connectivity) of $W_m(q)$ equals the degree of regularity $q$, for any $1 \leq m \leq q - 1$.

Shao, He and Shan [20] proved that in $W_m(q)$, $q = p^e$, $p$ prime, for $m \geq 2$, for any integer $l \neq 5,4 \leq l \leq 2p$ and any vertex $v$, there is a cycle of length $2l$ passing through the vertex $v$. We wish to remark that the edge-transitivity of $W_m(q)$ implies
the existence of a 2l cycle through any edge, a stronger statement. Li and Lih [12] used the Wenger graphs to determine the asymptotic behavior of the Ramsey number \( r_n(C_{2k}) = \Theta(n^{k/(k-1)}) \) when \( k \in \{2, 3, 5\} \) and \( n \to \infty \); the Ramsey number \( r_n(G) \) equals the minimum integer \( N \) such that in any edge-coloring of the complete graph \( K_N \) with \( n \) colors, there is a monochromatic \( G \). Representation (1) points to a relation of Wenger graphs with the moment curve \( t \mapsto (1, t, t^2, t^3, \ldots, t^m) \), and, hence, with the Vandermonde’s determinant, which was explicitly used in [25]. This is also in the background of some geometric constructions by Mellinger and Mubayi [17] of magnitude extremal graphs without short even cycles.

In Section 2, we determine the spectrum of the graphs \( W_m(q) \), defined as the multiset of the eigenvalues of the adjacency matrix of \( W_m(q) \). Futorny and Ustimenko [6] considered applications of Wenger graphs in cryptography and coding theory, as well as some generalizations. They also conjectured that the second largest eigenvalue \( \lambda_2 \) of the adjacency matrix of Wenger graphs \( W_m(q) \) is bounded from above by \( 2\sqrt{q} \). The results of this paper confirm the conjecture for \( m = 1 \) and \( 2 \), or \( m = 3 \) and \( q \geq 4 \), and refute it in other cases. We wish to point out that for \( m = 1 \) and \( 2 \), or \( m = 3 \) and \( q \geq 4 \), the upper bound \( 2\sqrt{q} \) also follows from the known values of \( \lambda_2 \) for the point-line \((q + 1)\)-regular incidence graphs of the generalized polygons \( PG(2, q) \), \( Q(4, q) \) and \( H(q) \) and eigenvalue interlacing (see Brouwer, Cohen and Neumaier [4]). In [13], Li, Lu and Wang showed that the graphs \( W_m(q) \), \( m = 1, 2 \), are Ramanujan, by computing the eigenvalues of another family of graph described by systems of linear equations in [10], \( D(k, q) \), for \( k = 2, 3 \). Their result follows from the fact that \( W_2(q) \simeq D(2, q) \), and \( W_2(q) \simeq D(3, q) \). For more on Ramanujan graphs, see Lubotzky, Phillips and Sarnak [15], or Murty [18]. Our results also imply that for fixed \( m \) and large \( q \), the Wenger graph \( W_m(q) \) are expanders. For more details on expanders and their applications, see Hoory, Linial and Wigderson [7], and references therein.

2. Main results

**Theorem 2.1.** For all prime power \( q \) and \( 1 \leq m \leq q - 1 \), the distinct eigenvalues of \( W_m(q) \) are

\[
\pm q, \pm \sqrt{mq}, \pm \sqrt{(m-1)q}, \ldots, \pm \sqrt{2q}, \pm \sqrt{q}, 0. \tag{2}
\]

The multiplicity of the eigenvalue \( \pm \sqrt{iq} \) of \( W_m(q) \), \( 0 \leq i \leq m \), is

\[
(q-1) q \sum_{d=i}^{m} \left(-1\right)^{d-i} \binom{q}{d-i} \binom{q-i}{k} q^{d-i-k}. \tag{3}
\]

**Proof.** As the graph \( W_m(q) \) is bipartite with partitions \( L \) and \( P \), we can arrange the rows and the columns of an adjacency matrix \( A \) of \( W_m(q) \) such that \( A \) has the following form:
\begin{equation}
A = \begin{pmatrix}
L & P \\
0 & N^T \\
N & 0
\end{pmatrix}
\end{equation}

which implies that
\begin{equation}
A^2 = \begin{pmatrix}
N^T N & 0 \\
0 & N N^T
\end{pmatrix}.
\end{equation}

As the matrices $N^T N$ and $NN^T$ have the same spectrum, we just need to compute the spectrum for one of these matrices. To determine the spectrum of $N^T N$, let $H$ denote the point-graph of $W_m(q)$ on $L$. This means that the vertex set of $H$ is $L$, and two distinct lines $[l]$ and $[l']$ of $W_m(q)$ are adjacent in $H$ if there exists a point $(p) \in P$, such that $[l] \sim (p) \sim [l']$ in $W_m(q)$. More precisely, $[l]$ and $[l']$ are adjacent in $H$, if there exists $p_1 \in \mathbb{F}_q$ such that for all $i = 1, \ldots, m$, we have

\begin{align*}
l_1 \neq l_1' & \quad \text{and} \quad l_{i+1} - l_1' = p_1 (l_i - l_i') \\
& \iff \quad l_1 \neq l_1' \quad \text{and} \quad l_{i+1} - l_1' = p_1 (l_i - l_i').
\end{align*}

This implies that $H$ is actually the Cayley graph of the additive group of the vector space $\mathbb{F}_q^{m+1}$ with a generating set
\begin{equation}
S = \{(t, tu, \ldots, tu^m) \mid t \in \mathbb{F}_q^*, u \in \mathbb{F}_q\}.
\end{equation}

Let $\omega$ be a complex $p$-th root of unity. For $x \in \mathbb{F}_q$, the trace of $x$ is defined as $tr(x) = \sum_{i=0}^{e-1} x^p$. The eigenvalues of $H$ are indexed after the $(m+1)$-tuples $(w_1, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, and can be represented in the following form (see Babai [1] and Lovász [14] for more details):

\begin{align*}
\lambda_{(w_1, \ldots, w_{m+1})} &= \sum_{(t, tu, \ldots, tu^m) \in S} \omega^{tr(tw_1)} \cdot \omega^{tr(tuw_2)} \cdot \ldots \cdot \omega^{tr(tuw_{m+1})} \\
&= \sum_{t \in \mathbb{F}_q^*, u \in \mathbb{F}_q} \omega^{tr(tw_1 + tw_2 + \cdots + tw_{m+1})} \\
&= \sum_{t \in \mathbb{F}_q^*, u \in \mathbb{F}_q^{m+1}} \omega^{tr(t(f(u)))} \quad \text{(where } f(u) := w_1 + w_2 u + \cdots + w_{m+1} u^m) \\
&= \sum_{t \in \mathbb{F}_q^*, f(u) = 0} \omega^{tr(t(f(u)))} + \sum_{t \in \mathbb{F}_q^*, f(u) \neq 0} \omega^{tr(t(f(u)))}.
\end{align*}

As $\sum_{t \in \mathbb{F}_q} \omega^{tr(tx)} = q - 1$ for $x = 0$, and $\sum_{t \in \mathbb{F}_q^*} \omega^{tr(tx)} = -1$ for every $x \in \mathbb{F}_q^*$, we obtain that
\begin{equation}
\lambda_{(w_1, \ldots, w_{m+1})} = \left| \{ u \in \mathbb{F}_q \mid f(u) = 0 \} \right| (q - 1) - \left| \{ u \in \mathbb{F}_q \mid f(u) \neq 0 \} \right|.
\end{equation}
Let $B$ be the adjacency matrix of $H$. Then $N^T N = B + qI$; this fact can be seen easily by examining the on- and off-diagonal entries of both sides of the equation. Therefore, the eigenvalues of $W_m(q)$ can be written in the form

$$\pm \sqrt{\lambda(w_1,\ldots,w_{m+1}) + q},$$

where $(w_1,\ldots,w_{m+1}) \in \mathbb{F}_q^{m+1}$. Let $f(X) = w_1 + w_2 X + \cdots + w_{m+1} X^m \in \mathbb{F}_q[X]$. We consider two cases.

1. $f = 0$. In this case, $|\{u \in \mathbb{F}_q \mid f(u) = 0\}| = q$, and $\lambda(w_1,\ldots,w_{m+1}) = q(q - 1)$. Thus, $W_m(q)$ has $\pm q$ as its eigenvalues.

2. $f \neq 0$. In this case, let $i = |\{u \in \mathbb{F}_q \mid f(u) = 0\}| \leq m$ as $1 \leq m \leq q - 1$. This shows that $\lambda(w_1,\ldots,w_{m+1}) = i(q - 1) - (q - i) = iq - q$ and implies that $\pm \sqrt{\lambda(w_1,\ldots,w_{m+1}) + q} = \pm \sqrt{iq}$ are eigenvalues of $W_m(q)$. Note that for any $0 \leq i \leq m$, there exists a polynomial $f$ over $\mathbb{F}_q$ of degree at most $m \leq q - 1$, which has exactly $i$ distinct roots in $\mathbb{F}_q$. For such $f$, $|\{u \in \mathbb{F}_q \mid f(u) = 0\}| = i$, and, hence, there exists $(w_1,\ldots,w_{m+1}) \in \mathbb{F}_q^{m+1}$, such that $\lambda(w_1,\ldots,w_{m+1}) = iq - q$. Thus, $W_m(q)$ has $\pm \sqrt{iq}$ as its eigenvalues, for any $0 \leq i \leq m$, and the first statement of the theorem is proven.

The arguments above imply that the multiplicity of the eigenvalue $\pm \sqrt{iq}$ of $W_m(q)$ equals the number of polynomials of degree at most $m$ (not necessarily monic) having exactly $i$ distinct roots in $\mathbb{F}_q$. To calculate these multiplicities, we need the following lemma. Particular cases of the lemma were considered in Zsigmondy [26], and in Cohen [5]. The complete result appears in A. Knopfmacher and J. Knopfmacher [8].

**Lemma 2.2.** (See [8].) Let $q$ be a prime power, and let $d$ and $i$ be integers such that $0 \leq i \leq d \leq q - 1$. Then the number $b(q,d,i)$ of monic polynomials in $\mathbb{F}_q[X]$ of degree $d$, having exactly $i$ distinct roots in $\mathbb{F}_q$ is given by

$$b(q,d,i) = \binom{q}{i} \sum_{k=0}^{d-i} (-1)^k \binom{q - i}{k} q^{d-i-k}. \quad (8)$$

By **Lemma 2.2**, the number of polynomials of degree at most $m$ in $\mathbb{F}_q[X]$ (not necessarily monic) having exactly $i$ distinct roots in $\mathbb{F}_q$ is

$$\sum_{d=i}^{m} (q - 1) b(q,d,i) = (q - 1) \binom{q}{i} \sum_{d=i}^{m} \sum_{k=0}^{d-i} (-1)^k \binom{q - i}{k} q^{d-i-k}. \quad (9)$$

This concludes the proof the theorem. \(\square\)

The previous result shows that $W_m(q)$ is connected and has $2m + 3$ distinct eigenvalues, for any $1 \leq m \leq q - 1$. As the diameter of a graph is strictly less than the number of
distinct eigenvalues (see [4, Section 4.1] for example), this implies that the diameter of Wenger graph is less or equal to $2m + 2$. This is actually the exact value of the diameter of the Wenger graph as shown by Viglione [23].

Since the sum of multiplicities of all eigenvalues of the graph $W_m(q)$ is equal to its order, and remembering that the multiplicity of $\pm q$ is one when $1 \leq m \leq q - 1$, we have a combinatorial proof of the following identity.

**Corollary 2.3.** For every prime power $q$, and every $m$, $1 \leq m \leq q - 1$,

$$\sum_{i=0}^{m} \binom{q}{i} \sum_{d=i}^{m} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k} = \frac{q^{m+1} - 1}{q - 1}. \quad (10)$$

The identity (10) seems to hold for all integers $q \geq 3$, so a direct proof is desirable. Other identities can be obtained by taking the higher moments of the eigenvalues of $W_m(q)$.

As we discussed in the introduction, for $m \geq q$, the graph $W_m(q)$ has $q^{m-q+1}$ components, each isomorphic to $W_{q-1}(q)$. This, together with Theorem 2.1, immediately implies the following.

**Proposition 2.4.** For $m \geq q$, the distinct eigenvalues of $W_m(q)$ are

$$\pm q, \pm \sqrt{(q-1)q}, \pm \sqrt{(q-2)q}, \ldots, \pm \sqrt{2q}, \pm \sqrt{q}, 0,$$

and the multiplicity of the eigenvalue $\pm \sqrt{iq}$, $0 \leq i \leq q - 1$, is

$$(q-1)q^{m+1-q} \binom{q}{i} \sum_{d=i}^{q} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k}.$$

3. Open questions

There are several open questions about the Wenger graphs $W_m(q)$ that we think are worth investigating: deciding whether these graphs are Hamiltonian, finding the lengths of all their cycles, determining their automorphism group,\(^1\) or determining the parameters of the linear codes whose Tanner graphs are the Wenger graphs.

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