On a Method in Algebra and Geometry*

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Often applications of a well-known fact or method are more exciting than the proof the fact itself and, as a result, the "theoretical" part of the problem is not paid sufficient attention, which is pedagogically wrong. To support this claim I shall introduce several problems in which good understanding of "why it works" is indispensable. The subject stems from a very simple problem.

1. *Solve the following system of equations:*

$$2x + 3y = 11, \quad 4x - 2y = -2. \tag{1}$$

Solution.

Multiplying both sides of the first equation by 2 and both sides of the second by 3, then adding the results and dividing by 16 we obtain $x = 1$ and, subsequently, $y = 3$. Therefore $\{(1,3)\}$ is the solution set of the system (1). □

How can one be sure that the new system obtained using the above method, namely $2x+3y = 11, x = 1$ is equivalent to (1), i.e. has the same set of solutions as (1)? Using a graphical interpretation of (1), the question can be restated as "why do lines whose

*This version contains some minor improvements of the original text which appeared in the *The Amatyc Review* published by the American Mathematical Association for two-year colleges: Vol. 12, Number 1/2, Fall 1990/Spring 1991, p. 70-72/47-48.

equations are $4x - 2y = -2$ and $x = 1$ intersect the line whose equation is $2x + 3y = 11$ at the same point?"

The method we are using above is familiar to everyone. Despite its importance, I find very few books of algebra, precalculus or (even!) linear algebra in which it is proven. When I want to prove it in class, the typical reaction from students is that they have known the fact from the cradle and have serious doubts that it requires any proof at all. Nevertheless, it is important for understanding of what follows and we prove the following:

Theorem *Let $F_1(x, y)$, $F_2(x, y)$ be two algebraic expressions defined on a set D from $\mathbb{R}^2$. Let $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. Then the following two systems*

$$\text{(I)} \quad F_1(x, y) = 0, \qquad F_2(x, y) = 0; \text{ and}$$

$$\text{(II)} \qquad F_1(x, y) = 0, \qquad \alpha F_1(x, y) + \beta F_2(x, y) = 0$$

are equivalent.

Proof To prove that (I) and (II) are equivalent is to show that each solution of (I) is a solution of (II) and vice versa. If both solution sets are empty, the statement obviously holds. If at least one of the solution sets is not empty, then the following argument shows that neither is the second one and they are equal:

(1) If (a, b) is a solution of (I), then $F_1(a, b) = 0$, $F_2(a, b) = 0$ and, as a result, $\alpha F_1(a, b) + \beta F_2(a, b) = 0 + 0 = 0$, which implies that (a, b) is also a solution of (II);

(2) If (a, b) satisfies (II) and $\beta \neq 0$, then the relations $F_1(a, b) = 0$ and $\alpha F_1(a, b) + \beta F_2(a, b) = 0$ imply $\alpha \cdot 0 + \beta F_2(a, b) = 0$, which is equivalent to $F_2(a, b) = 0$, and this completes the proof. □

In terms of the graphical interpretation of (I) and (II), the theorem says that the curves defined by the equations $F_2(x, y) = 0$ and $\alpha F_1(x, y) + \beta F_2(x, y) = 0$ intersect the graph of the curve with the equation $F_1(x, y) = 0$ at the same set of points (Figure 1).

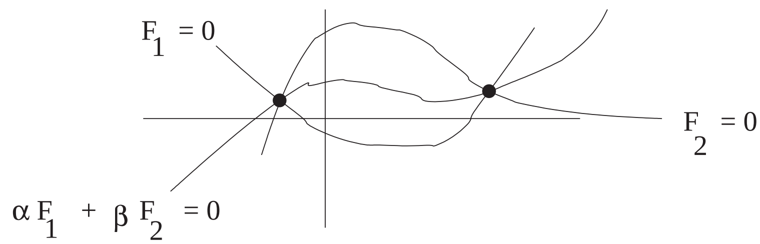

Figure 1

The examples that follow are intended to demonstrate how some well-known rather non-trivial problems of the algebraic/geometric nature can be easily handled by the theorem above, without lengthy manipulations. Some problems allow further generalizations which we will not discuss here. In order for my students to fully appreciate the idea, I have included these problems in the homework prior to the lecture, and students were allowed to use any techniques they wished.

2. *Find an equation of the line passing through the points of intersections of two circles C_1 and C_2 given respectively by the equations*

$$x^2 + (y-3)^2 = 9 \quad \textit{and} \quad (x-4)^2 + (y-2)^2 = 16. \tag{2}$$

Solution.

First of all we observe that the circles do intersect, because the distance between the centers is $\sqrt{17}$ and the radii are 3 and 4. Therefore, system (2) has a solution.

Replacing the second equation in (2) by its difference with the first one, we obtain system (3) which, according to the Theorem, is equivalent to (2) ($\alpha = 1,\ \beta = -1$):

$$x^2 + (y-3)^2 = 9, \quad 4x - y = 2. \tag{3}$$

The line having equation $4x - y = 2$ is the one we are looking for, because it crosses C_1 at the same points as C_2. $\square$

Many students do not realize this fact and continue with the solution of (3) by substituting $y = 4x - 2$ into the first equation, finding the coordinates of points of intersections of the circles, and finally finding an equation of the line passing through these two points.

3. *Let C_1, C_2, C_3 be three circles in the plane, such that any two of them intersect at two points. Prove that three common chords are concurrent, i.e., intersect at one point (Figure 2).*

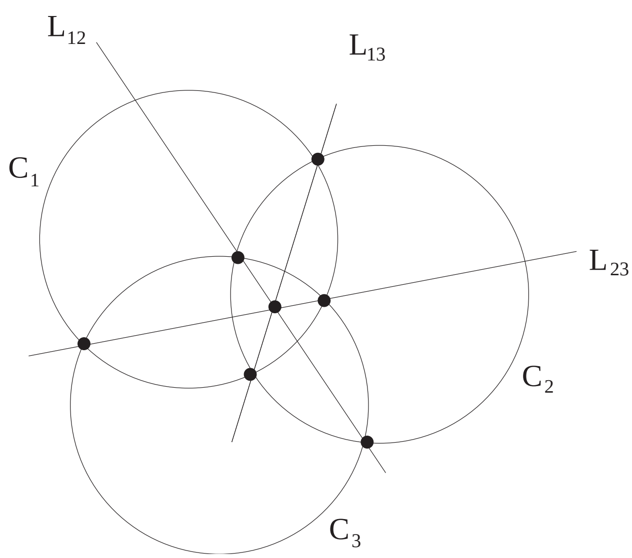

Figure 2

Proof.

Let $F_i(x, y) = (x - a_i)^2 + (y - b_i)^2 - (r_i)^2 = 0$ be an equation of C_i in a Cartesian coordinate system, $i = 1, 2, 3$. Let L_{ij} be the line passing through the points of intersection of C_i and C_j. Using the same argument as in Problem 2, we conclude that $L_{ij}(x, y) = F_i(x, y) - F_j(x, y) = (a_j - a_i)x + (b_j - b_i)y + C_{ij} = 0$ is an equation of L_{ij}, where C_{ij} is the constant term. The coordinates of the point of intersection of lines L_{12} and L_{13} can be found by solving the system

$$L_{12}(x,y) = F_1(x,y) - F_2(x,y) = 0, \quad L_{13}(x,y) = F_1(x,y) - F_3(x,y) = 0. \tag{4}$$

Replacing the second equation of (4) by its difference with the first one we obtain:

$$F_1(x,y) - F_2(x,y) = 0, \quad F_3(x,y) - F_2(x,y) = 0, \tag{5}$$

which, according to the Theorem ($\alpha = 1,\ \beta = -1$) is equivalent to (4). But the second equation in (5) is an equation of the line $L_{32}(= L_{23})$! Therefore the lines L_{13} and L_{23} intersect the line L_{12} at the same point. $\square$

4. *(a) Let E_1 and E_2 be two ellipses whose axes are mutually parallel. Show that if the ellipses intersect at four points, then the four points lie on a circle (Figure 3).*

   (b) Let P_1 and P_2 be two parabolas whose axes are perpendicular. Show that if the parabolas intersect at four points, then the four points lie on a circle.

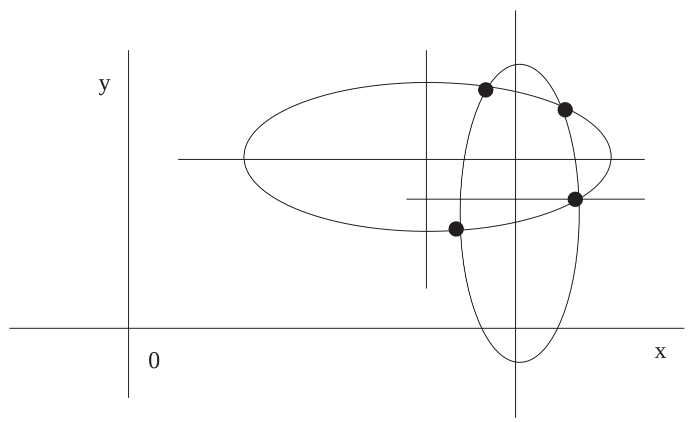

Figure 3

Solution.

(a) In a Cartesian coordinate system with axes parallel to those of ellipses, the coordinates of the points of intersection are solutions of the system:

$$\begin{aligned} F_1(x,y) &= \frac{(x-a_1)^2}{c_1^2} + \frac{(y-b_1)^2}{d_1^2} - 1 = 0 \\ F_2(x,y) &= \frac{(x-a_2)^2}{c_2^2} + \frac{(y-b_2)^2}{d_2^2} - 1 = 0. \end{aligned} \tag{6}$$

In order to show that the points of intersection lie on a circle we replace the second equation in system (1) by the linear combination $\alpha F_1(x,y)+\beta F_2(x,y)=0$ where α and β are chosen so (see below) that the equation $\alpha F_1(x,y)+\beta F_2(x,y)=0$ is an equation of a circle. Then, the result follows immediately from the Theorem. Since $Ax^2+By^2+Cx+Dy+F=0$ is a circle if and only if $A=B\neq 0$ and it is not a point or an empty set, α and β can be found by equating the coefficients at x^2 and y^2 in $\alpha F_1(x,y)+\beta F_2(x,y)=0$. Hence,

$$\frac{\alpha}{c_1^2}+\frac{\beta}{c_2^2}=\frac{\alpha}{d_1^2}+\frac{\beta}{d_2^2} \tag{7}$$

If $|c_1|=|d_1$, then the ellipse E_1 is a circle and the statement (a) is proven. If $c_1|\neq|d_1|$, set $\beta=1$ in (2) and solve (2) for α. This completes the proof. □

(b) In a Cartesian coordinate system with axes parallel to those of parabolas, P_1 and P_2 have equations in the form:

$$\begin{aligned} F_1(x,y) &= a_1x^2+b_1x+c_1-y=0 \quad \text{and} \\ F_2(x,y) &= a_2y^2+b_2y+c_2-x=0, \quad \text{respectively.} \end{aligned}$$

Since $a_1\neq 0$ and $a_2\neq 0$, the equation $a_2F_1(x,y)+a_1F_2(x,y)=0$ represents the equation of the circle passing through the points of intersection of the parabolas. □

The following problem was an exercise in the original article:

5. *Given an $\angle CBD$ and a point A in its exterior. Let rays AR_i, $i=1,2,\ldots,n$, intersect side BD at a point C_i and side BD at a point D_i. Let $Q_{ij}=Q_{ji}$ be the intersection of the segments C_iD_j and C_jD_i, $i,j\in\{1,2,\ldots,n\}$, $i\neq j$. Prove that all points Q_{ij}'s are colinear.*

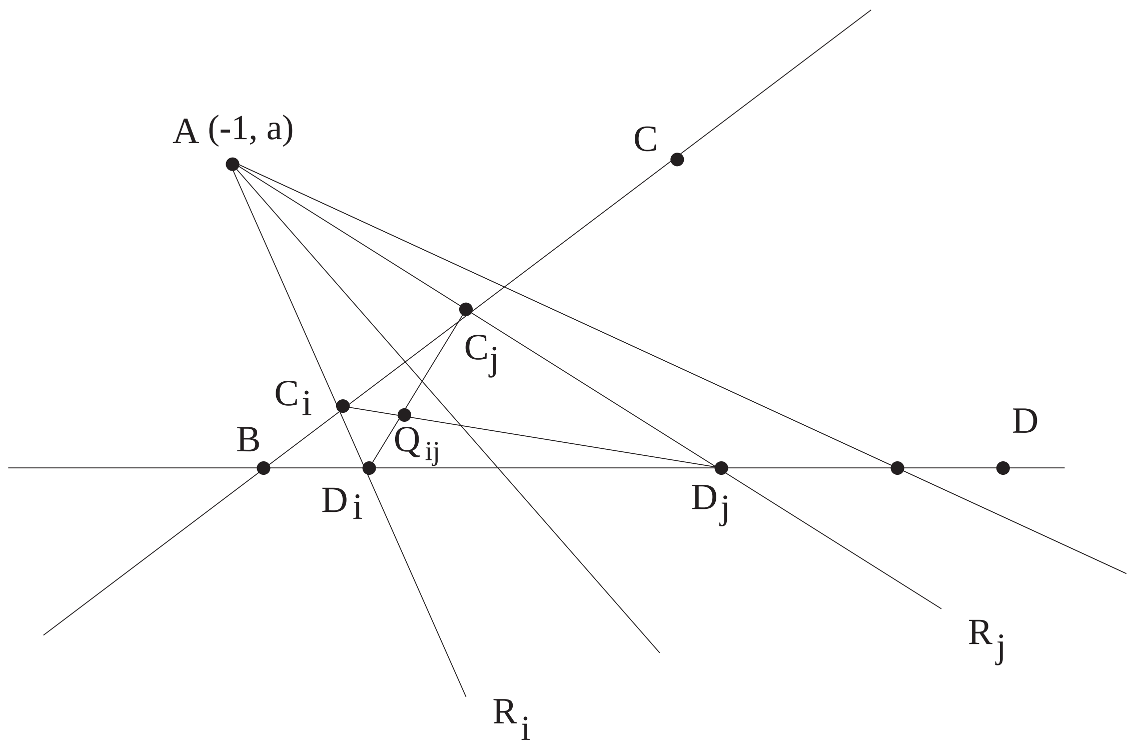

Figure 4

Solution.

Choose a coordinate system such that B is the origin, and lines BD and BC become x-axis and y-axis respectively (thus, it is not necessarily a rectangular). Assume that point A and ray BC lie on the same side of the line BD. Choose a scale such that point A has coordinates $(-1, a)$. Denote the slope of the line AC_i by m_i (it exists, since the lines AC_i and BC are not parallel). Notice that $m_i \neq 0$, and $m_i \neq m_j$ for $i \neq j$. Then an equation of the line AC_i can be written in the form: $y - a = m_i(x + 1)$. Therefore, the points C_i and D_j have coordinates $(0, m_i + a)$ and $(-1 - 1/m_j, 0)$, respectively, and the coordinates of the point Q_{ij} can be found by solving the following system of equations of the lines c_iD_j and C_jD_i:

$$\frac{x}{-1 - a/m_j} + \frac{y}{m_i + a} = 1$$

$$\frac{x}{-1 - a/m_i} + \frac{y}{m_j + a} = 1$$

Subtracting these equations and simplifying the result, we obtain $y = ax$. Therefore, all points Q_{ij}s lie on the line with the equation $y = ax$. □