New Upper Bounds for the Greatest Number of Proper Colorings of a \((V,E)\)-Graph

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ABSTRACT

Let \(\mathcal{G}\) denote the family of simple undirected graphs on \(v\) vertices having \(e\) edges (\(v,e\)-graphs) and \(P(G;\lambda)\) be the chromatic polynomial of a graph \(G\). For the given integers \(v, e,\) and \(\lambda\), let \(f(v, e, \lambda)\) denote the greatest number of proper colorings in \(\lambda\) or less colors that a \((v,e)\)-graph \(G\) can have, i.e., \(f(v, e, \lambda) = \max\{P(G; \lambda); G \in \mathcal{G}\}\). In this paper we determine some new upper bounds for \(f(v, e, \lambda)\).

1. INTRODUCTION

The definitions in this paper are based on [3]. All graphs we consider are undirected labeled graphs without loops and multiple edges. Let \(V(G)\) and \(E(G)\) denote a set of vertices and edges of \(G\), respectively. The number of elements of a finite set \(A\) is denoted by \(|A|\). We write \(v = v(G) = |V(G)|\) and \(e = e(G) = |E(G)|\). By \(c = c(G)\) we denote the number of connected components of graph \(G\). For any positive integer \(\lambda\), a proper \(\lambda\)-coloring of a labeled graph \(G\) is a mapping of \(V(G)\) into the set \(\{1, 2, \ldots, \lambda\}\) (the set of colors) such that no two adjacent vertices of \(G\) have the same image. The chromatic number of a graph \(G\), denoted \(\chi(G)\), is the least \(\lambda\) (number of colors) for which there exists a proper coloring of \(G\). Let \(P(\lambda) = P(G; \lambda)\) denote the number of proper \(\lambda\)-colorings of \(G\). This function was introduced in [2] and turned out to be a polynomial function of \(\lambda\).

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Let $\mathcal{F} = \mathcal{F}_n$ be a family of all graphs having $v$ vertices and $e$ edges ($v, e$-graphs). Let $\lambda$ be the number of colors. Denote by $f(v, e, \lambda)$ the greatest number of proper $\lambda$-colorings that a $(v, e)$-graph can have, i.e., $f(v, e, \lambda) = \min \{ \# \text{of } H: H \in \mathcal{F} \}$. In this paper we find some new nontrivial upper bounds for $f(v, e, \lambda)$ in the general case, i.e., in the case when the only restrictions on the integers $v, e$, and $\lambda$ are $0 \leq e \leq v(v - 1)/2$, $\lambda \geq 2$.

The main result is the following:

**Theorem 1.1.** Let $v, e$, and $\lambda$ be integers, $0 \leq e \leq v(v - 1)/2$, $\lambda \geq 2$. Let $f(v, e, \lambda)$ be equal to the greatest number of proper $\lambda$-colorings of a graph with $v$ vertices and $e$ edges. Then

$$f(v, e, \lambda) \leq A \lambda^e,$$

where $A$ is the least of the following three quantities:

$$\left(1 - \frac{1}{\lambda}\right)^{\left\lfloor \frac{v(v-1)}{2} - e \right\rfloor}, \quad \left(1 - \frac{e}{\lambda} + \left(\frac{e}{\lambda}\right)^2\right)^{\frac{\lambda - 1}{\lambda - 1 + e}} \quad \text{or} \quad \frac{\lambda - 1}{\lambda - 1 + e}.$$

The question was motivated by the analysis of the running time of the backtrack algorithm for the graph coloring problem (see Wilf [10]; Bender and Wilf [1]). Another source of related problems is a paper of Wright [11], where an asymptotic approximation to the number $M_{v, \lambda}$, the total number of proper $\lambda$-colorings of all $(v, e)$-graphs found for a fixed $\lambda$, large $v$, and all $e$. Problems similar to ours but for different families of graphs (all graphs on $v$ vertices whose chromatic number is equal to $k$) were considered by Tomassen [7,8]. Several other instances of the problem were considered by the author in [4], [5].

2. PROOF OF THEOREM 1.1.

The inequality

$$f(v, e, \lambda) \leq \left(1 - \frac{1}{\lambda}\right)^{\left\lfloor \frac{v(v-1)}{2} - e \right\rfloor} \lambda^e$$

was proved in [4]. In order to get other upper bounds, we apply some known facts based on the famous Inclusion—Exclusion Principle, or Sieve Method, and the corresponding interpretation of the chromatic polynomial of a graph due to Whitney [9]. Here we give a brief list of the corresponding facts. All proofs can be found in Lovasz [8, III, §2].

**Proposition 2.1.**

(i) (Inclusion—Exclusion Formula). Let $A_1, \ldots, A_n$ be arbitrary events of a probability space $(\Omega, P)$. For each $I \subseteq \{1, \ldots, n\}$, let
\[ A_\lambda = \prod_{i=d}^{n} A_i; \quad A_\lambda = \Omega; \]

and let

\[ \sigma_i = \sum_{i=1}^{n} P(A_i), \quad \sigma_0 = 1. \]

Then

\[ P(\overline{A_1}, \overline{A_2}, \ldots, \overline{A_j}) = \sum_{k=0}^{n} (-1)^{k+1} \sigma_k, \quad (2.2) \]

(ii) (Bonferroni Inequalities). The partial sums of

\[ P(\overline{A_1}, \overline{A_2}) = \sigma_0 + \sigma_1 - \sigma_2 + \ldots, \quad (2.3) \]

are alternating in sign.

(iii) (Selberg's Sieve, particular case). If the events \( A_i, 1 \leq i \leq n, \) are pair-wise independent and \( P(A_i) = \rho \) for all \( i, 1 \leq i \leq n, \) then

\[ P(\overline{A_1}, \ldots, \overline{A_j}) \leq \frac{1}{1 + ap/(1 - \rho)}. \quad (2.4) \]

Given a labeled graph \( G \) with \( v \) vertices and \( e \) edges and an integer \( \lambda \geq 1, \) we associate with \( G \) the sample space \( \Omega \) as follows:

\[ \Omega = \{ \text{all (proper and improper) colorings of } G \text{ in } \lambda \text{ colors} \}. \]

Let \( V(G) = \{1, 2, \ldots, v\} \) and \( E(G) = \{ (i, j), (i, j), \ldots, (i, j) \} \). For each \( k, 1 \leq k \leq e, \) we define the event \( A_k \) as

\[ A_k = \{ \omega \in \Omega \mid \text{vertices } i_k \text{ and } j_k \text{ are colored in the same color} \}. \]

Obviously, \( |\Omega| = \lambda^v \) and for each \( k, 1 \leq k \leq e, \) \( |A_k| = \lambda \cdot \lambda^{v-k} = \lambda^{v-k} \) (a color for \( i_k \) and \( j_k \) can be chosen in \( \lambda \) different ways and each of the remaining \( v-2 \) vertices can be colored in \( \lambda \) colors independently of each other).

We define a probability function \( P \) on \( \Omega \) in the usual way, assuming that the probability of each elementary event is equal to \( 1/|\Omega| \). Then

\[ P(A_k) = \lambda^{v-k}/\lambda^v = 1/\lambda, \quad 1 \leq k \leq e. \quad (2.5) \]

A coloring of \( G \) is proper if and only if none of the events \( A_k, 1 \leq k \leq e, \) happens. Therefore

\[ P(G; \lambda) = P(\overline{A_1}, \ldots, \overline{A_j}) \cdot \lambda^* \]
and the results of Proposition 2.1 can be used in order to get upper bounds for \( P(G; \lambda) \). But the following questions should be answered first: what is the probability of the intersection of two events \( A_i \) and \( A_j \) for \( i \neq j \), and are they pairwise independent?

The event \( A_i \cdot A_j \) takes place if and only if vertices \( i, j \) are colored in the same color and vertices \( i, j \) are colored in the same color. Therefore

\[
[A_i \cdot A_j] = \begin{cases} \\
1 - \frac{\lambda \cdot (1 - \lambda)^{n-1}}{\lambda^2}, & \text{if } \{i, j\} \cap \{1, 2, \ldots, n\} = \emptyset \\
\lambda - \lambda^2, & \text{if } |\{i, j\} \cap \{1, 2, \ldots, n\}| = 1.
\end{cases}
\]

Thus

\[
[A_i \cdot A_j] = \lambda^{n-1} \quad \text{and} \quad P(A_i \cdot A_j) = \lambda^{n-1}/\lambda^n = 1/\lambda^2
\] (2.6)

for all \( k, 1 \leq k < \lambda < \infty \). Using (2.5) and (2.6) we get

\[
P(A_k \cdot A_j) = P(A_i \cdot A_j) \cdot P(A_k),
\]

which implies the independence of \( \Lambda \) and \( A_i \) for all \( i, j, 1 \leq i < j \leq \lambda \).

For our model,

\[
P(A_i) = \frac{1}{\lambda}, \quad \sigma_1 = \sum_{i=1}^{\lambda} P(A_i) = \frac{\lambda}{\lambda},
\]

\[
\sigma_2 = \sum_{i=1}^{\lambda} P(A_i \cdot A_j) = \frac{e(\lambda - 1)}{2\lambda}.
\]

Using this and multiplying both sides of (2.3) and (2.4) by \( \|E\| \), we obtain two upper bounds for \( P(G; \lambda) \) and \( f(v, e, \lambda) \): from (2.3):

\[
P(G; \lambda) \leq f(v, e, \lambda) < \left(1 - \frac{e}{\lambda} + \left(\frac{e}{\lambda^2}\right)\right)\lambda^n
\] (2.7)

from (2.3):

\[
P(G; \lambda) \leq f(v, e, \lambda) \leq \frac{\lambda - 1}{\lambda - 1 + e}\lambda^n.
\] (2.8)

Combining (2.1), (2.7), and (2.8), we obtain a proof of Theorem 1.1. For some ranges of parameters, the comparison of (2.1), (2.7), and (2.8) is simple, and it gives

\[
\text{for } e = 0, 1, \lambda + 1.
\]
the right sides of (2.7) and (2.8) are equal;

$$\text{for } e > \max \left\{ \lambda + 1, (\lambda - 1) \left[ \frac{\lambda}{\lambda - 1} \right]^{s-1} - 1 \right\},$$

the bound given by (2.8) is better than those given by (2.1) and (2.7);

$$\text{for } \nu - 1 \leq e \leq \lambda + 1,$$

the bound given by (2.1) is better than ones in (2.7) and (2.8).

References


