

On the nondegeneracy of quadratic forms

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In everything I seek to grasp the fundamental...

... ..
The crux, the roots, the inmost hearts...

—Boris Pasternak¹

1 Introduction

Undoubtedly, every reader has tried to clarify a notion in one source by consulting another, only to be frustrated that the presentations are inconsistent in vocabulary or notation. Recently, this happened to us in a study of conics. While reading Peter Cameron’s *Combinatorics: Topics, Techniques, and Algorithms* [6], we encountered a definition of a non-singular quadratic form $Q = Q(x, y, z)$ as one that

...cannot be transformed into a form in less than three variables by any non-singular linear substitution of the variables x, y, z .

Though the definition was clear, it did not appear to translate into simple criteria for determining whether a given quadratic form was singular. In searching for such a test, we found that various sources used the word “singular” to describe quadratic forms in (what seemed to be) completely different ways. Complicating matters further was that terms such as “degenerate” and “reducible” started to appear, and were often used interchangeably with “singular.”

In all, we found five statements in the literature. While we believed these statements to be equivalent, few sources proved the equivalence of even two of them, and only two sources proved the equivalence of three. This prompted us to extend these statements to n dimensions (some of these extensions we did not see in the literature), and write what we believe are clear, straightforward, and self-contained proofs that they are in fact equivalent. We found the process both enjoyable and instructive, and we hope the reader will find what we present to be valuable.

In particular, many of the proofs we used draw ideas from the basic principles of analysis, algebra, linear algebra, and geometry. We think that some of these

¹Appeared in 1956. Translated from the Russian by Avril Pyman.

equivalences can serve as useful exercises in related undergraduate courses, as they help to stress the unity of mathematics.

The main object of our study below will be a quadratic form and its associated quadric; we will now define these terms. Additional definitions and related results can be found in Hoffman and Kunze [12] or in Shilov [22], for example. Let \mathbb{F} be a field whose characteristic, denoted $\text{char}(\mathbb{F})$, is odd or zero. Examples include the fields of rational numbers, real numbers, and complex numbers, as well as finite fields containing an odd (prime power) number of elements. Now, let \mathbb{F}^n denote the n -dimensional vector space over \mathbb{F} . Any $(n - 1)$ -dimensional subspace of \mathbb{F}^n is called a *hyperplane*. By $\mathbb{F}[X_1, \dots, X_n]$, we will denote the ring of polynomials with (commuting) indeterminants X_1, \dots, X_n , and coefficients in \mathbb{F} . It will often be convenient to view a polynomial of k indeterminants as a polynomial of one of them, with coefficients being polynomials of the other $k - 1$ indeterminants. For instance, a polynomial in $\mathbb{F}[X_1, X_2, X_3]$ may be viewed as an element of $\mathbb{F}[X_2, X_3][X_1]$; i.e. a polynomial of X_1 whose coefficients are polynomials of X_2 and X_3 .

A *quadratic form* is a polynomial

$$Q = Q(X_1, \dots, X_n) = \sum_{1 \leq i, j \leq n} a_{ij} X_i X_j, \quad (1)$$

where the coefficients $a_{ij} \in \mathbb{F}$ are such that $a_{ij} = a_{ji}$ for all i, j . Alternatively, Q defines a polynomial function $Q : \mathbb{F}^n \rightarrow \mathbb{F}$, where $(\alpha_1, \dots, \alpha_n) \mapsto Q(\alpha_1, \dots, \alpha_n)$. Note that we use the same notation Q for both the algebraic object (i.e. the polynomial) and the corresponding function. The *quadric* corresponding to Q is defined as

$$\mathbb{A}(Q) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n : Q(\alpha_1, \dots, \alpha_n) = 0\}.$$

Quadrics are a special case of *affine varieties* (hence the “ \mathbb{A} ” in $\mathbb{A}(Q)$). $\mathbb{A}(Q)$ is also called the *graph* of Q in \mathbb{F}^n .

Let V denote the set of all degree one polynomials of X_1, \dots, X_n over \mathbb{F} having zero constant term, called linear forms, together with the zero polynomial. V is clearly a vector space over \mathbb{F} , and $\{X_1, \dots, X_n\}$ is a basis of V . It is well known that the null space of every non-zero linear form f is a hyperplane W , and we say that f *defines* W . Conversely, every hyperplane is the null space of a non-zero linear form. Two forms define the same hyperplane if and only if they are non-zero scalar multiples of one another.

For any linear transformation $\varphi : V \rightarrow V$ and any quadratic form Q , we can substitute $\varphi(X_i)$ for X_i in Q for all $i = 1, \dots, n$. After simplifying the result by combining like terms, we again obtain a quadratic form, which we denote by $\tilde{Q} = Q(\varphi(X_1), \dots, \varphi(X_n))$ or $\tilde{Q}(X_1, \dots, X_n)$.

We will sometimes wish to evaluate this new form at a point $T = (t_1, \dots, t_n)$. To do this, note that $\varphi(X_1) = a_1 X_1 + \dots + a_n X_n$ for some $a_1, \dots, a_n \in \mathbb{F}$. Let $\varphi(X_1)|_T = a_1 t_1 + \dots + a_n t_n$, and define $\varphi(X_2)|_T, \dots, \varphi(X_n)|_T$ similarly. Then the evaluation of $Q(\varphi(X_1), \dots, \varphi(X_n))$ at T is simply $Q(\varphi(X_1)|_T, \dots, \varphi(X_n)|_T)$.

Next, a comment on derivatives. We treat partial derivatives formally, as is done in algebra. For example, to differentiate Q with respect to X_1 , we view Q as an element of $\mathbb{F}[X_2, \dots, X_n][X_1]$:

$$Q = a_{11}X_1^2 + (2a_{12}X_2 + \dots + 2a_{1n}X_n)X_1 + \sum_{1 < i, j \leq n} a_{ij}X_iX_j$$

and thus

$$\frac{\partial Q}{\partial X_1} = 2a_{11}X_1 + (2a_{12}X_2 + \dots + 2a_{1n}X_n).$$

Partial derivatives with respect to the other indeterminants are defined similarly. We can therefore define the gradient vector of Q , evaluated at a point $T \in \mathbb{F}^n$, as $\nabla Q(T) = \left(\frac{\partial Q}{\partial X_1}(T), \dots, \frac{\partial Q}{\partial X_n}(T) \right)$.

In the following, \mathbb{K} will always denote a field that is either equal to \mathbb{F} , or is a quadratic extension $\mathbb{F}(m)$ of \mathbb{F} for some $m \in \mathbb{K} \setminus \mathbb{F}$ such that $m^2 \in \mathbb{F}$. The element m , and thus the field \mathbb{K} , may change depending on the context.

The goal of this paper is to state and prove the following theorem.

Theorem. Let $n \geq 2$, $\text{char}(\mathbb{F}) \neq 2$, Q be a non-zero quadratic form over \mathbb{F} as defined by (1), and $\mathbb{A}(Q)$ be the corresponding quadric. Furthermore, let $r = 1$ or 2 . Then the following statements are equivalent.

1. The $n \times n$ matrix $M_Q := (a_{ij})$ of the quadratic form Q has rank r .
2. Q is a product of two linear forms with coefficients in \mathbb{K} . These forms are scalar multiples of one another if $r = 1$, and are not if $r = 2$.
3. There exists a non-singular linear transformation $\varphi : V \rightarrow V$ such that $\tilde{Q} = Q(\varphi(X_1), \dots, \varphi(X_n))$ contains precisely r of the indeterminants X_1, \dots, X_n ; furthermore, for any other non-singular linear transformation, this number is at least r .
4. If $r = 1$, $\mathbb{A}(Q)$ is a hyperplane. If $r = 2$, $\mathbb{A}(Q)$ is the union of two distinct hyperplanes.
5. The vector space $\mathcal{N} := \{T \in \mathbb{F}^n : \nabla Q(T) = 0\}$ has dimension $n - r$.

Note that statements 1, 3, and 4 primarily use terms of linear algebra, with statement 4 having a geometric flavor. Statement 2 is algebraic, while statement 5 relates to analysis.

Let R^t denote the transpose of a matrix R , and let $X = (X_1, \dots, X_n)^t$ be the column vector of indeterminants. Then, in terms of matrix multiplication, $[Q] = X^t M_Q X$. From now on, we will view the 1×1 matrix $[Q]$ as the polynomial Q , and simply write $Q = X^t M_Q X$.

Let $\varphi : V \rightarrow V$ be a linear transformation, $Y_i = \varphi(X_i)$ for all i , and $Y = (Y_1, \dots, Y_n)^t$. Then we define

$$\varphi(X) = (\varphi(X_1), \dots, \varphi(X_n))^t,$$

and therefore

$$\varphi(X) = (Y_1, \dots, Y_n)^t = Y.$$

Letting B_φ be the matrix of φ with respect to a basis $\{X_1, \dots, X_n\}$ of V ,

$$Y = B_\varphi X.$$

This allows us to use matrix multiplication in order to determine $M_{\tilde{Q}}$. Indeed, we have

$$\tilde{Q} = \tilde{Q}(X_1, \dots, X_n) = X^t M_{\tilde{Q}} X$$

and

$$\begin{aligned} \tilde{Q} &= Q(\varphi(X_1), \dots, \varphi(X_n)) = Q(Y_1, \dots, Y_n) \\ &= Y^t M_Q Y \\ &= (B_\varphi X)^t M_Q (B_\varphi X) \\ &= X^t (B_\varphi^t M_Q B_\varphi) X. \end{aligned}$$

The equality $X^t M_{\tilde{Q}} X = X^t (B_\varphi^t M_Q B_\varphi) X$, viewed as an equality of 1×1 matrices with polynomial entries, implies that $M_{\tilde{Q}} = B_\varphi^t M_Q B_\varphi$.

Recall that for any square matrix M and any non-singular matrix N of the same dimensions, $\text{rank}(MN) = \text{rank}(M) = \text{rank}(NM)$. As φ is non-singular, so is B_φ , and we have $\text{rank}(M_Q) = \text{rank}(M_{\tilde{Q}})$.

Finally, we remind the reader of the fundamental fact that given any quadratic form Q , there exists a non-singular linear transformation ψ of V such that if $Q' := Q(\psi(X_1), \dots, \psi(X_n))$, $M_{Q'}$ is diagonal. In other words,

$$Q' = d_1 X_1^2 + \dots + d_n X_n^2,$$

where all $d_i \in \mathbb{F}$.

Before proving the theorem, we will illustrate it with an example.

Example. Let $n = 4$ and $\mathbb{F} = \mathbb{R}$ (the example is also valid whenever $\text{char}(\mathbb{F}) \neq 2$). Consider the quadratic form

$$Q = 2X_3^2 + 2X_1X_2 - X_1X_3 + 3X_1X_4 - 4X_2X_3 - 6X_3X_4.$$

Let us check that Q satisfies each of the five statements from the theorem.

1. As

$$M_Q = \begin{pmatrix} 0 & 1 & -1/2 & 3/2 \\ 1 & 0 & -2 & 0 \\ -1/2 & -2 & 2 & -3 \\ 3/2 & 0 & -3 & 0 \end{pmatrix},$$

one can use row reduction to verify that $r = \text{rank}(M_Q) = 2$.

2. $Q = (X_1 - 2X_3)(2X_2 - X_3 + 3X_4)$, the product of $r = 2$ polynomials that are not scalar multiples of each other.

3. Define a linear transformation $\varphi : V \rightarrow V$ by

$$\begin{aligned}\varphi(X_1) &= X_1 + 2X_3 \\ \varphi(X_2) &= \frac{1}{2}X_2 + \frac{1}{2}X_3 - \frac{3}{2}X_4 \\ \varphi(X_3) &= X_3 \\ \varphi(X_4) &= X_4.\end{aligned}$$

It is a straightforward verification that φ is non-singular, and that

$$\tilde{Q} = Q(\varphi(X_1), \varphi(X_2), \varphi(X_3), \varphi(X_4)) = X_1X_2,$$

which contains only $r = 2$ indeterminants. Note that if there existed a non-singular linear transformation $\phi : V \rightarrow V$ such that $\hat{Q} = Q(\phi(X_1), \dots, \phi(X_n))$ had only one indeterminant, then $1 = \text{rank}(M_{\hat{Q}}) = \text{rank}(M_Q) = 2$, a contradiction.

4. It is clear from the above factorization of Q that $\mathbb{A}(Q)$ is the union of two hyperplanes whose equations are $X_1 - 2X_3 = 0$ and $2X_2 - X_3 + 3X_4 = 0$. As $X_1 - 2X_3$ and $2X_2 - X_3 + 3X_4$ are not scalar multiples of one another, these hyperplanes are distinct.

5.

$$\nabla Q = \begin{pmatrix} \partial Q / \partial X_1 \\ \partial Q / \partial X_2 \\ \partial Q / \partial X_3 \\ \partial Q / \partial X_4 \end{pmatrix} = \begin{pmatrix} 2X_2 - X_3 + 3X_1 \\ 2X_1 - 4X_3 \\ -X_1 - 4X_2 + 4X_3 - 6X_4 \\ 3X_1 - 6X_3 \end{pmatrix} = 2M_Q X.$$

Therefore, \mathcal{N} is the null space of the matrix $2M_Q$, which has dimension $2 = 4 - 2 = n - r$, as claimed.

2 Proof of Main Result

We are now ready to prove our theorem.

1 \Rightarrow 2: Suppose M_Q has rank $r = 1$ or 2 . As mentioned above, there exists a non-singular transformation ψ of V such that $Q' = Q(\psi(X_1), \dots, \psi(X_n)) = d_1X_1^2 + \dots + d_nX_n^2$ with all d_i in \mathbb{F} . Recall that $\text{rank}(M_{Q'}) = \text{rank}(M_Q)$. If $r = 1$, we permute the indices as necessary to conclude that $Q' = d_1X_1^2$, where d_1 is non-zero. Then $Q = d_1(\psi^{-1}(X_1))^2$ factors as claimed. If instead $r = 2$, then permuting indices as necessary implies that $Q' = d_1X_1^2 + d_2X_2^2$, where $d_1 \neq 0$ and $d_2 \neq 0$. Let $m \in \mathbb{K}$ such that $m^2 = -d_2/d_1$. Then

$$\begin{aligned}Q' &= d_1X_1^2 + d_2X_2^2 \\ &= d_1 \left(X_1^2 - \frac{-d_2}{d_1} X_2^2 \right) \\ &= d_1 (X_1^2 - m^2 X_2^2) \\ &= (d_1X_1 - d_1mX_2)(X_1 + mX_2).\end{aligned}$$

Clearly, the two factors of Q' are independent in V ; otherwise, $m = -m$, which is equivalent to $m = 0$ because $\text{char}(\mathbb{F}) \neq 2$. This implies $d_2 = 0$, a contradiction. Hence,

$$Q(X) = \psi^{-1}(d_1X_1 - d_1mX_2) \cdot \psi^{-1}(X_1 + mX_2);$$

since ψ^{-1} is non-singular, the factors of Q are independent.

2 \Rightarrow (1 and 3): Suppose Q factors over \mathbb{K} . If $r = 1$, then $Q = a(a_1X_1 + \dots + a_nX_n)^2$ with $a, a_i \in \mathbb{K}$ and $a \neq 0$. Permuting indices as necessary, suppose $a_1 \neq 0$; then apply the linear substitution φ defined by

$$\varphi(X_1) = \frac{1}{a_1}X_1 - \sum_{i=2}^n \frac{a_i}{a_1}X_i \text{ and } \varphi(X_i) = X_i \text{ for all } i = 2, \dots, n.$$

This transformation is non-singular, and the resulting quadratic form $\tilde{Q} = aX_1^2$ has $r = 1$ indeterminant, implying statement 3. Furthermore, $\text{rank}(M_Q) = \text{rank}(M_{\tilde{Q}}) = 1$ implies statement 1.

If instead $r = 2$, then $Q = (a_1X_1 + \dots + a_nX_n)(b_1X_1 + \dots + b_nX_n)$ with $a_i, b_i \in \mathbb{K}$ such that the factors do not differ by a scalar multiple. This is equivalent to the vectors (a_1, \dots, a_n) and (b_1, \dots, b_n) being linearly independent. Therefore, permuting indices as necessary, the vectors $(a_1, a_2), (b_1, b_2) \in \mathbb{F}^2$ are linearly independent with $a_1 \neq 0$ and $b_2 \neq 0$. Apply the linear substitution φ defined by

$$\begin{aligned} \varphi(X_1) &= \alpha_{11}X_1 + \alpha_{12}X_2 + \dots + \alpha_{1n}X_n \\ \varphi(X_2) &= \alpha_{21}X_1 + \alpha_{22}X_2 + \dots + \alpha_{2n}X_n \\ \varphi(X_i) &= X_i \text{ for all } i = 3, \dots, n, \end{aligned}$$

where (α_{ij}) is the inverse of the matrix

$$\left(\begin{array}{cccc|c} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{array} \right),$$

and I is the $(n-2) \times (n-2)$ identity matrix. This non-singular transformation produces the quadratic form $\tilde{Q} = X_1X_2$, which contains $r = 2$ indeterminants. Note that if there existed a linear transformation $\phi : V \rightarrow V$ such that $\hat{Q} := Q(\phi(X_1), \dots, \phi(X_n))$ had only one indeterminant, then $1 = \text{rank}(M_{\hat{Q}}) = \text{rank}(M_Q) = \text{rank}(M_{\tilde{Q}}) = 2$, a contradiction. This proves statement 3. Statement 1 follows because $\text{rank}(M_Q) = \text{rank}(M_{\tilde{Q}}) = 2$.

3 \Rightarrow 4: First, we assume $r = 1$ and that there exists non-singular $\varphi : V \rightarrow V$ such that \tilde{Q} contains only one indeterminant. Then $\tilde{Q} = aX_1^2$, and so $Q = a\varphi^{-1}(X_1)^2$. Therefore, $\mathbb{A}(Q)$ is the hyperplane $\varphi^{-1}(X_1) = 0$.

Alternatively, suppose that $r = 2$ and that \tilde{Q} contains exactly two indeterminants, say X_1 and X_2 . Then $\tilde{Q} = aX_1^2 + bX_1X_2 + cX_2^2$. Let $m^2 = b^2 - 4ac$. Then $\tilde{Q} = (a_1X_1 + a_2X_2)(b_1X_1 + b_2X_2)$, where $a_i, b_i \in \mathbb{K} = \mathbb{F}(m)$; thus, $Q = \varphi^{-1}(a_1X_1 + a_2X_2) \cdot \varphi^{-1}(b_1X_1 + b_2X_2)$. Note that if $\varphi^{-1}(a_1X_1 + a_2X_2)$ and $\varphi^{-1}(b_1X_1 + b_2X_2)$ were scalar multiple of one another, then the non-singular linear transformation $\phi : V \rightarrow V$ defined such that

$$\begin{aligned}\phi(X_1) &= \frac{1}{a_1}\varphi(X_1) - \frac{a_2}{a_1}X_2 \\ \phi(X_i) &= X_i \text{ for all } i = 2, \dots, n\end{aligned}$$

would produce a quadratic form in only one variable, contradicting statement 3. Therefore, $\mathbb{A}(Q)$ is the union of the distinct hyperplanes $\varphi^{-1}(a_1X_1 + a_2X_2) = 0$ and $\varphi^{-1}(b_1X_1 + b_2X_2) = 0$.

4 \Rightarrow 2: Let $a_1X_1 + \dots + a_nX_n = 0$ be an equation of a hyperplane W of \mathbb{K}^n such that $W \subseteq \mathbb{A}(Q)$. Then for every solution $(\alpha_1, \dots, \alpha_n)$ of $a_1\alpha_1 + \dots + a_n\alpha_n = 0$, $Q(\alpha_1, \dots, \alpha_n) = 0$. As not all a_i are zero, we may assume by permuting the indices as necessary that $a_1 \neq 0$. Dividing by a_1 , we rewrite the equation $a_1X_1 + \dots + a_nX_n = 0$ as $X_1 + a'_2X_2 + \dots + a'_nX_n = 0$. Viewing Q as an element of $\mathbb{K}[X_2, \dots, X_n][X_1]$ and dividing it by $X_1 + (a'_2X_2 + \dots + a'_nX_n)$ with remainder, we obtain

$$Q = q \cdot (X_1 + (a'_2X_2 + \dots + a'_nX_n)) + r, \quad (2)$$

with quotient $q \in \mathbb{K}[X_2, \dots, X_n][X_1]$ and remainder $r \in \mathbb{K}[X_2, \dots, X_n]$. Now, $Q(w_1, \dots, w_n) = 0 = w_1 + (a'_2w_2 + \dots + a'_nw_n)$ for every $(w_1, \dots, w_n) \in W$. Furthermore, for every $(w_2, \dots, w_n) \in \mathbb{F}^{n-1}$, there exists $w_1 \in \mathbb{F}$ such that $(w_1, \dots, w_n) \in W$. This implies that $r(w_2, \dots, w_n) = 0$ for every $(w_2, \dots, w_n) \in \mathbb{F}^{n-1}$. The following lemma will allow us to conclude that $r = 0$:

Lemma. Let $n \geq 1$, $\text{char}(\mathbb{F}) \neq 2$, and $f \in \mathbb{F}[X_1, \dots, X_n]$ be a quadratic form that vanishes on \mathbb{F}^n . Then $f = 0$.

Proof. We proceed by induction. For $n = 1$, $f = f(X_1) = aX_1^2$ for some $a \in \mathbb{F}$. Then $0 = f(1) = a$, and so $f = 0$.

Suppose the statement holds for all quadratic forms containing fewer than n indeterminants. We write f as

$$f(X_1, \dots, X_n) = aX_1^2 + f_2(X_2, \dots, X_n)X_1 + f_3(X_2, \dots, X_n),$$

where $a \in \mathbb{F}$, $f_2 \in \mathbb{K}[X_2, \dots, X_n]$ is a linear form, and $f_3 \in \mathbb{K}[X_2, \dots, X_n]$ is a quadratic form. As $\text{char}(\mathbb{F}) \neq 2$, 0 , 1 , and -1 are distinct elements of \mathbb{F} . As f vanishes on \mathbb{F}^n , f vanishes at every point of the form $(\alpha, \alpha_2, \dots, \alpha_n) \in \mathbb{F}^n$ with $\alpha \in \{0, 1, -1\}$. Therefore, we obtain

$$0 = f(0, \alpha_2, \dots, \alpha_n) = f_3(\alpha_2, \dots, \alpha_n) \quad (3)$$

$$0 = f(1, \alpha_2, \dots, \alpha_n) = a + f_2(\alpha_2, \dots, \alpha_n) + f_3(\alpha_2, \dots, \alpha_n) \quad (4)$$

$$0 = f(-1, \alpha_2, \dots, \alpha_n) = a - f_2(\alpha_2, \dots, \alpha_n) + f_3(\alpha_2, \dots, \alpha_n) \quad (5)$$

for all $(\alpha_2, \dots, \alpha_n) \in \mathbb{F}^{n-1}$. By induction hypothesis, (3) implies that $f_3 = 0$. Then (4) and (5) imply that $a = 0$ and that $f_2(\alpha_2, \dots, \alpha_n) = 0$ for all $(\alpha_2, \dots, \alpha_n) \in \mathbb{F}^{n-1}$. Let $f_2 = a_2X_2 + \dots + a_nX_n$. If $f_2 \neq 0$, then there exists $a_i \neq 0$. Setting $\alpha_i = 1$ and $\alpha_j = 0$ for all $j \neq i$, we obtain $0 = f_2(0, \dots, 0, 1, 0, \dots, 0) = a_i \neq 0$, a contradiction. Thus, $f_2 = 0$, and so $f = 0$. \square

Indeed, this lemma implies that $r = 0$. Therefore, Q factors into a product of two linear forms, each defining a hyperplane in \mathbb{K}^n . We now proceed based on whether $r = 1$ or 2. If $r = 1$, then $\mathbb{A}(Q)$ is a hyperplane of \mathbb{K}^n . Thus, both factors of Q must define $\mathbb{A}(Q)$, and so they are non-zero scalar multiples of one another. If instead $r = 2$, then $\mathbb{A}(Q)$ is the union of two hyperplanes of \mathbb{K}^n , corresponding to the two factors of Q . As the hyperplanes are distinct, the factors are not scalar multiples of one another.

1 \Leftrightarrow 5: Suppose $\text{rank}(M_Q) = r$. Let $\text{null}(M_Q)$ denote the dimension of the null space of M_Q , and so $\text{null}(M_Q) = n - r$. It is straightforward to verify that $(2M_Q)(T) = \nabla Q(T)$ for all $T \in \mathbb{F}^n$. Thus, since $\text{char}(\mathbb{F}) \neq 2$, $\dim(\mathcal{N}) = \text{null}(2M_Q) = \text{null}(M_Q) = n - r$.

Conversely, $n - r = \dim(\mathcal{N}) = \text{null}(2M_Q) = \text{null}(M_Q)$; so $\text{rank}(M_Q) = r$.

This concludes our proof of the theorem. \square

This theorem justifies the following definition:

Definition. A quadratic form $Q = Q(X_1, \dots, X_n)$ over a field \mathbb{F} , $\text{char}(\mathbb{F}) \neq 2$, is called *degenerate*, *reducible*, or *singular* if it satisfies any of the five statements of the theorem. Otherwise, Q is called *non-degenerate*, *irreducible*, or *non-singular*.

3 Table of vocabulary usage

We conclude this paper by presenting a table of resources that utilize the statements from the theorem. While some discuss forms (often in the $n = 3$ case), others consider conics in the classical projective plane $PG(2, q)$. As conics are simply quadrics in projective space, it is not surprising that the characterizations of degenerate quadrics and degenerate conics are nearly identical. Thus, we will not make any further attempt to distinguish them. In addition, we found many books that discuss equivalent notions, but not in the context of quadratic forms or conics. Such sources have therefore been excluded from this table.

Abbreviations used in the table are as follows: D for degenerate, S for singular, and R for reducible. The columns labeled Prop. 1 to Prop. 5 refer to the five properties from the theorem. Note that each of the five properties is referred to by multiple names, and every name is used to describe multiple properties! Entries with a * indicate that the resource mentions the property, but does not use a particular word to describe it. We use ** to indicate that

in [25], the descriptive terms “proper line pair” or “repeated line” are used to describe property 4.

	Prop. 1	Prop. 2	Prop. 3	Prop. 4	Prop. 5
[1]			D or S		
[2]		*			R
[3]		D	*	D	
[4]	S				
[5]				R	
[6]			S		
[7]	*	D			
[8]	*	R		R	S
[9]	S		*		
[11]	S				
[10]	*		*		S
[13]	*			*	*
[14]	D				
[15]	*	D		D	
[16]	*				S
[17]	*	D		D	
[18]		*		R	
[19]	D				
[20]	*	D or R		*	*
[21]	D		*		
[22]	S				
[23]	*			D	
[24]	*	R			S
[25]		*		**	

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