
**GRAVITATIONAL FORCE OF A UNIFORM BALL**

This document is a modification of two notes:

one by Jun Li [http://math.stanford.edu/~jli/Math52/Gravity.pdf](http://math.stanford.edu/~jli/Math52/Gravity.pdf)

and

another by Felix Lazebnik: “On Earth, Moon and Maple ”.

When we calculate the gravity force or electric force exerted by a spacial object or a spacial charge, it is often convenient to replace the the object by a point mass, or a point charge. It was first done by Isaac Newton (1642 - 1727), when he studied motions of celestial bodies, in particular the motion of the Moon around the Earth. We consider the following

**Question:** Given a ball of radius $R$ of uniform density $\delta$, and a point mass $m$ of distance $a$ from the center of the ball, assuming the gravitational constant is $G$, find the gravitational force of the ball exerted on the point mass.

Our first try is to move the origin to the center of the ball, and set the point mass at $(0, 0, a)$, $a > 0$. Then by symmetry, the force on the point mass is pointing downwards. Let $M$ denote the mass of the ball, so

$$M = \frac{4}{3} \pi R^3 \delta$$

Let $\Delta V$ be a “tiny bit” at $(x, y, z)$, of volume $\Delta V$. The gravitational force of this bit on the point mass at $(0, 0, a)$ is

$$\vec{F} = G(\Delta V \delta) m \frac{x \vec{i} + y \vec{j} + (z - a) \vec{k}}{(x^2 + y^2 + (z - a)^2)^{3/2}},$$

just use the vector form of the Gravity Law.

Summing the force vectors for all tiny bit of the ball and taking the limit, we obtain, using the symmetry, that the resulting force is directed along $z$ axis and its magnitude is

$$F = G \delta m \int \int \int \frac{(z - a)}{(x^2 + y^2 + (z - a)^2)^{3/2}} dV$$

where the triple integral is taken over the ball $x^2 + y^2 + z^2 \leq R^2$.

It is straightforward to set up the iterated integral using the spherical coordinate to evaluate it:

$$F = G \delta m \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R} \frac{\rho \cos \phi - a}{(\rho^2 \sin^2 \phi + (\rho \cos \phi - a)^2)^{3/2}} \rho^2 \sin \phi d\rho d\phi d\theta$$
However, finding the answer is relatively tricky due to the denominator.

Instead, we set the point mass at the origin and set the ball be centered at \((0, 0, a)\). This way, the magnitude of the force is expressed as

\[
F = G\delta m \int \int \int \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \, dV
\]

where the triple integral is taken over the ball \(x^2 + y^2 + (z - a)^2 \leq R^2\).

To find the integral limit, we need to find the equation of the circle centered at \((0, 0, a)\) of radius \(R\) in the \(yz\)-plane. (Because it is rotational symmetric (in \(\theta\)), we only need to look at the \(yz\)-plane.) The equation of the circle resulting from intersecting the ball with the \(yz\)-plane is given (in the \(yz\)-plane) by

\[
y^2 + (z - a)^2 = R^2,
\]

or, using \(\rho\) and \(\phi\), by

\[
(\rho \sin \phi)^2 + (\rho \cos \phi - a)^2 = R^2.
\]

Solving it, we get

\[
\rho_{\pm}(\phi) = a \cos \phi \pm \sqrt{a^2 \cos^2 \phi + (R^2 - a^2)}.
\]

**Case A:** \(0 < R \leq a\). This is the case when the point mass is not inside the ball.

In this case, \(\phi\) is at most \(\arcsin(R/a) \leq \pi/2\), \(R^2 - a^2 \leq 0\), and so both values of \(\rho_{-}(\phi)\) and \(\rho_{+}(\phi)\) are nonnegative. Hence, the integration region can be described as

\[
0 \leq \theta \leq 2\pi, \quad \rho_{-}(\phi) \leq \rho \leq \rho_{+}(\phi), \quad 0 \leq \phi \leq \alpha.
\]

Thus \(\sin \alpha = R/a\), and

\[
F = G\delta m \int_0^{2\pi} \int_0^\alpha \int_{\rho_{-}(\phi)}^{\rho_{+}(\phi)} \frac{\rho \cos \phi}{\rho^3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta =
\]

\[
G\delta m \int_0^{2\pi} \int_0^\alpha \sqrt{a^2 \cos^2 \phi + (R^2 - a^2)} \cos \phi \sin \phi \, d\phi \, d\theta =
\]

(Using the substitution \(u = \sqrt{a^2 \cos^2 \phi + (R^2 - a^2)}\))

\[
Gm\left(\frac{4}{3}\pi R^3\delta\right) \cdot \frac{1}{a^2} = G\frac{mM}{a^2}.
\]

In conclusion,

**the force is the same as when we replace the ball by a point mass with mass equal to that of the ball!**
**Case B:** $R > a > 0$. This is the case when the point mass is inside the ball.

In this case, $\rho_- (\phi) < 0$ and $\rho_+ (\phi) \geq 0$. Hence, the integration region can be described as

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \rho \leq \rho_+ (\phi), \quad 0 \leq \phi \leq \pi.$$ 

Therefore

$$F = G \delta m \int_0^{2\pi} \int_0^\pi \int_0^{\rho_+ (\phi)} \frac{\rho \cos \phi}{\rho^3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta =$$

$$G \delta m \int_0^{2\pi} \int_0^\pi \int_0^{\rho_+ (\phi)} (a \cos \phi + \sqrt{a^2 \cos^2 \phi + (R^2 - a^2)}) \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta =$$

$$2\pi G \delta m \left( \int_0^\pi a \cos^2 \phi \sin \phi \, d\phi + \int_0^\pi \sqrt{a^2 \cos^2 \phi + (R^2 - a^2)} \cos \phi \sin \phi \, d\phi \right) =$$

(using the substitution $u = \sqrt{a^2 \cos^2 \phi + (R^2 - a^2)}$)

$$G m \left( \frac{4}{3} \pi a^3 \delta \right) \cdot \frac{1}{a^2} = G \frac{mM_a}{a^2},$$

where $M_a = \frac{4}{3} \pi a^3 \delta$ is the mass of the concentric ball of radius $a$.

Hence, in this case,

**the force is the same as from a concentric ball of radius $a$!**

Using additivity of triple integral, in this case, we can represent the triple integral over the ball $x^2 + y^2 + (z - a)^2 \leq R^2$ as the sum of two triple integrals: one over the ball $x^2 + y^2 + (z - a)^2 \leq a^2$ and another over the spherical layer $a^2 \leq x^2 + y^2 + (z - a)^2 \leq R^2$. The result above shows that the spherical layer exerts zero gravitational force on the origin. So we have an important corollary:

**a spherical layer of uniform density exerts zero gravitational force at its interior points!**

This is one of my (F.L.) favorite mathematical result which illustrates the superiority of a technique over the intuition. If the exponent of the distance in the denominator of the Law of Gravity were 2.0000001 or 1.9999999, the result would be different. (Another example is the series $1 + 1/2^p + 1/3^p + \ldots$ in the neighborhood of $p = 1$.)

Mentioning intuition, I (F.L.), of course mean my own intuition and intuition of several other mortals... Things change if we include people of genius. To Newton the fact above was quite obvious. His argument can be found in many places on the web and in many books. Look for *Newton’s Shell Theorem*, as Newton considers the set of concentric shells of uniform density (hollow spheres) whose union is a ball or a spherical layer, and first
proved the result for such shells. One can also look for the potential of a solid ball.

Comment 1. Everywhere above we assumed that the density was uniform. It is really not true for the Earth. The density of the crust (surface layer) is known to be about $1/2$ of the average density (which is about $5.5$ grams per cubic centimeter), and the density of the inner core is about $3$ times greater than the average density. Another thing is that the shape of the Earth is not exactly a ball....

Comment 2. Similar results hold in electrostatics for a charged sphere. It is mentioned in The Feynman Lectures on Physics (Book 5), by R. P. Feynman, R.B. Leighton and M. Sands, Addison-Wesley 1964, that it might be Benjamin Franklin who was first to point out that the field inside the charged sphere is zero. He mentioned this to Joseph Priestley in 1767, who conjectured that the electric force may vary inversely proportional to the distance. At the time the similarity between the Law of Gravity and the Coulomb’s law was not obvious at all. Coulomb’s law was stated only 18 years later (published in 1785) after he confirmed it experimentally, and the Gauss’s law (from which both results follow) – much later.