H6.1 Find an equation of a surface consisting of all points $P$ for which the distance from $P$ to the $x$-axis is twice the distance from $P$ to the $yz$-plane. Identify the surface.

Solution. Let $P : (x, y, z)$. Then, by considering a diagram, one concludes that the distance from $P$ to the $x$-axis is $\sqrt{y^2 + z^2}$, and the distance from $P$ to the $yz$-plane is $|x|$. Thus we have:

$$\sqrt{y^2 + z^2} = 2|x|.$$ 

As two sides of this equation are nonnegative, it is equivalent to

$$4x^2 - y^2 - z^2 = 0 \text{ or } x^2 = \frac{y^2}{2^2} + \frac{z^2}{2^2}.$$ 

The surface is a cone, whose axis is the $x$-axis. Traces perpendicular to $x$-axis are circles. Traces by the planes through $x$-axes are intersecting lines (any plane through $x$-axis has equation $By + Cz = 0$ for some constants $B$ and $C$. Solve the system of equations $x^2 = \frac{y^2}{2^2} + \frac{z^2}{2^2}$ and $By + Cz = 0$). Planes orthogonal to either $y$-axis or $z$-axis, cut the surface by hyperbolas.

If one has difficulty seeing why the distances are as we claim, one could use the formuli we studied: one for the distance from a point to a line, and another for a distance from a point to a plane...

H6.2 A rectangular box stands on horizontal floor. A ball rolls off a top of the box with a (horizontal) speed of 3 ft/s directed perpendicular to the edge. The box is 4 ft high.

(a) How far from the vertical wall of the box will the ball fall on the floor?

(b) What is the speed of the ball at the instant of impact?

(c) Find the angle between the trajectory of the ball and the vertical line drawn through the point of impact.

(d) Suppose the ball rebounds from the floor at the same angle with which it hits the floor, but loses 10% of its speed due to energy loss on impact. Where does the ball strike the floor on the second impact?

Solution. The whole motion takes place in a plane. We chose a coordinate system such that the point where the ball leaves the surface of the box has coordinate $(0, 4)$ (moment $t = 0$), and $x$-axis is parallel to the initial velocity vector. That is

(a) We know that $\vec{v}(t) = \vec{v}_0 - gt\vec{j} = \langle 3, -gt \rangle$, where $g = 32.17$ ft/s$^2$ (9.8 m/s$^2$). Therefore the position at moment $t$,

$$\vec{r}(t) = \vec{r}(0) + 3t\vec{i} - (1/2)gt^2\vec{j} = \langle 0, 4 \rangle + \langle 3t, -(1/2)gt^2 \rangle = \langle 3t, 4 - (1/2)gt^2 \rangle.$$ 

The ball hits the floor when $4 - (1/2)gt^2 = 0$. Since $t > 0$, it happens at $t_1 = \sqrt{8/g} \approx 0.4987$ seconds. At this moment $x$-coordinate is $d_1 = 3\sqrt{8/g} \approx 1.496$ ft.

(b) The velocity at the moment of impact is $\vec{v}(t_1) = \langle 3, -gt_1 \rangle$. Hence, the speed $|\vec{v}(t_1)| = \sqrt{9 + 8g} \approx 16.32$.

(c) The slope of the curve is $dy/dx = (dy/dt) / (dx/dt) = -gt/3$, so at the moment $t_1$ it is $-gt_1/3 \approx -5.347$. (Could also get it by considering the right triangle with legs equal to the absolute
values of the components of the velocity vector at the moment of impact.) Therefore the acute angle between the velocity and positive direction of the $x$-axis is $\arctan(-5.347) \approx -1.3859 \approx -79.41^\circ$. Therefore the acute angle with the vertical line has measure $90 - 79.41 \approx 10.59^\circ$.

(d) The ball rebounds with the speed $v_2 = 0.10\sqrt{9 + 8g} \approx 1.632$ ft/s. The inclination angle is $\alpha \approx 79.41^\circ$ degrees. We have computed in class that the horizontal distance it will travel until the next impact with the floor is $d_2 = \frac{v_2^2}{g} \sin 2\alpha$. Therefore $d_2 \approx 2.4234$. Therefore the corresponding $x$-coordinate is $d_1 + d_2 \approx 3.92$. Hence, the second impact happens at 3.92 ft from the box.