

Asymptotic Model of a Nonlinear Adaptive Visco-elastic Rod

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Abstract

Hegedus and Cowin [2, 3] considered an adaptive elasticity model for bone remodeling in 1970s. Based on Hegedus and Cowin's model, I. Figueiredo and L. Trabucho [5] recently studied a asymptotic model of a adaptive elastic rod. Recent studies have suggested that dissipation of acoustic energy is not due primarily to the viscous properties of interstitial blood and marrow, but rather the trabeculae. Based on this we consider the trabeculae to be modeled as a Kelvin-Voigt viscoelastic solid. We apply the asymptotic expansion method as developed by Figueiredo and Trabucho to obtain a nonlinear adaptive visco-elastic rod model. It is shown that a similar analysis holds in the Kelvin-Voigt viscoelastic case.

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1 Introduction

In a series of papers Hegedus and Cowin [2, 3] developed a theory of thermomechanical adaptive elasticity. In a second paper these results were specialized to the isothermal quasi-static case. As the general reader may not be acquainted with this theory we summarize it here. Cowin assumes that the bulk density ρ of the porous body may be written in the form $\rho = \gamma v$, where γ is the density of the material forming the matrix and v is the volume fraction of material present. The motion of the material, expressed in Eulerian coordinates is given in the form $x_i = \Xi_i(\mathbf{X}, t)$, where $\mathbf{X} = (X_k)$ are the reference (Lagrangian) coordinates. The gradient of the deformation tensor is given in the form

$$F_{ik} = \frac{\partial x_i}{\partial X_j}.$$

Let the volume fraction of the reference coordinates be denoted by ξ ; then

$$\xi = v [\det \mathbf{F} + \det \dot{\mathbf{F}}].$$

If $\mathbf{T} := (T_{ij})$ is the stress tensor and $\mathbf{b} := (b_i)$ are the body forces then quasistatic equilibrium is specified as [6]

$$T_{ij,j} + \gamma V b_i = 0.$$

The constitutive equation for stress is given in terms of the specific free energy, Ψ which we propose to depend on $\dot{\mathbf{F}}$ also, as

$$T_{ij} = \gamma v \left[\frac{\partial \Psi}{\partial F_{ik}} F_{jk} + \frac{\partial \Psi}{\partial \dot{F}_{ik}} \dot{F}_{jk} \right];$$

whereas the conservation of mass is expressed in the form

$$\dot{\xi} = \frac{c}{\gamma} [\det \mathbf{F} + \det \dot{\mathbf{F}}],$$

where c is the rate at which mass per unit volume is added or subtracted from the porous matrix structure. We now consider the restriction to small strains. The right Cauchy-Green dilation tensor may be defined in terms of the gradient of the deformation tensor as

$$\mathbf{C} := \mathbf{F}^T \mathbf{F};$$

whereas the right Cauchy-Green dilation tensor is defined as

$$\mathbf{B} := \mathbf{F} \mathbf{F}^T$$

and the Green-Lagrange strain tensor becomes

$$E = \frac{1}{2}(C - I).$$

It is convenient to use displacement $\mathbf{u} := \mathbf{x} - \mathbf{X}$. We denote ξ_0 as a reference volume fraction and

$$\phi = \xi - \xi_0$$

as the change in volume fraction from the reference volume fraction.

In terms of the displacement the tensors \mathbf{F} , \mathbf{C} , \mathbf{E} , take the forms [6]

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u},$$

$$\mathbf{C} = \mathbf{F}\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u}^T \nabla \mathbf{u},$$

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u}^T \nabla \mathbf{u}).$$

The constitutive relation for the stress tensor then takes on the form

$$T_{ij} = \frac{\gamma(\xi_0 + \phi)}{(\det(\mathbf{I} + 2\mathbf{E}))^{\frac{1}{2}}} \left[F_{im} F_{jp} \frac{\partial \Psi}{\partial E_{mp}} + \dot{F}_{im} \dot{F}_{jp} \frac{\partial \Psi}{\partial \dot{E}_{mp}} \right].$$

The rate of change of mass equation may now be written as

$$\dot{\phi} = \frac{c}{\gamma} \left[(\mathbf{I} + 2\mathbf{E})^{\frac{1}{2}} + (\mathbf{I} + 2\dot{\mathbf{E}})^{\frac{1}{2}} \right].$$

If the displacement gradients are small we can linearize the theory leading to the strain tensor taking on the form

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

By using the constitutive relations we may then truncate, to second order terms, the remodeling equation may be written as

$$\begin{aligned} \dot{\phi}^\epsilon &= a^\epsilon(\phi^\epsilon) + A_{kl}^\epsilon(\phi^\epsilon) e_{kl}^\epsilon(u^\epsilon) + \bar{A}_{kl}^\epsilon(\phi^\epsilon) e_{kl}^\epsilon(\dot{u}^\epsilon) \\ &+ B_{ijkl}^\epsilon(\phi^\epsilon) e_{ij}^\epsilon(u^\epsilon) e_{kl}^\epsilon(u^\epsilon) + \bar{B}_{ijkl}^\epsilon(\phi^\epsilon) e_{ij}^\epsilon(\dot{u}^\epsilon) e_{kl}^\epsilon(\dot{u}^\epsilon). \end{aligned}$$

2 the Nonlinear Adaptive Visco-elastic Rod Problem

We adopt the same notation as in [5] and [7]. Let ω be an open, bounded, connected subset of \mathbf{R}^2 . We study a cylindrical rod which we identify with a solid occupying the reference configuration

$$\bar{\Omega}^\epsilon = \omega^\epsilon \times [0, L] \quad (2.1)$$

where ϵ is very small with respect to the length of the rod L , $\omega^\epsilon = \epsilon\omega$.

Let

$$\Gamma^\epsilon = \partial\omega^\epsilon \times (0, L), \Gamma_0^\epsilon = \bar{\omega}^\epsilon \times \{0\}, \Gamma_L^\epsilon = \bar{\omega}^\epsilon \times \{L\}, \quad (2.2)$$

An arbitrary point of $\bar{\Omega}^\epsilon$ will be denoted by $x^\epsilon = (x_1^\epsilon, x_2^\epsilon, x_3^\epsilon)$ the unit outer normal vector to the boundary by $\mathbf{n}^\epsilon = (n_i^\epsilon)$ and the differential operators $\partial/\partial x_i^\epsilon$ and $n_i^\epsilon(\partial/\partial x_i^\epsilon)$ by ∂_i^ϵ and ∂_n^ϵ respectively.

The three-dimensional problem is stated as follows.

Find $u^\epsilon = (u_i^\epsilon) : \bar{\Omega}^\epsilon \times [0, T] \rightarrow R^3, \phi^\epsilon : \bar{\Omega}^\epsilon \times [0, T] \rightarrow R$ such that:

Equilibrium Equation

$$-\partial_j^\epsilon \sigma_{ij} = \gamma(\xi_0 + P_\eta(\phi^\epsilon))f_i^\epsilon, \quad (2.3)$$

Constitutive equation (Kelvin-Voigt):

$$\begin{aligned} \sigma_{ij}^\epsilon &= c_{ijkl}^\epsilon(\phi^\epsilon)e_{kl}^\epsilon(u^\epsilon) + d_{ijkl}^\epsilon(\phi^\epsilon)e_{kl}^\epsilon(\dot{u}^\epsilon) \\ c_{ijkl}^\epsilon &= (\xi_0 + M_\rho \cdot P_\eta(\phi^\epsilon))a_{ijkl}^\epsilon(M_\rho \cdot P_\eta(\phi^\epsilon)), \\ d_{ijkl}^\epsilon &= (\xi_0 + M_\rho \cdot P_\eta(\phi^\epsilon))b_{ijkl}^\epsilon(M_\rho \cdot P_\eta(\phi^\epsilon)), \\ e^\epsilon &= \frac{1}{2}(\partial_j^\epsilon u_i^\epsilon + \partial_i^\epsilon u_j^\epsilon) \end{aligned} \quad (2.4)$$

Boundary Conditions (Pure Traction)

$$\begin{aligned} \sigma_{ij}^\epsilon n_j^\epsilon &= g_i^\epsilon, \quad \text{in } \Gamma^\epsilon \times (0, T), \\ \sigma_{ij}^\epsilon n_j^\epsilon &= h_i^\epsilon, \quad \text{in } (\Gamma_0^\epsilon \cup \Gamma_L^\epsilon) \times (0, T) \end{aligned} \quad (2.5)$$

Remodeling Rate Equation

$$\begin{aligned} \dot{\phi}^\epsilon &= a^\epsilon(\phi^\epsilon) + A_{kl}^\epsilon(\phi^\epsilon)e_{kl}^\epsilon(u^\epsilon) + \bar{A}_{kl}^\epsilon(\phi^\epsilon)e_{kl}^\epsilon(\dot{u}^\epsilon) \\ &+ B_{ijkl}^\epsilon(\phi^\epsilon)e_{ij}^\epsilon(u^\epsilon)e_{kl}^\epsilon(u^\epsilon) + \bar{B}_{ijkl}^\epsilon(\phi^\epsilon)e_{ij}^\epsilon(\dot{u}^\epsilon)e_{kl}^\epsilon(\dot{u}^\epsilon), \quad \text{in } \Omega^\epsilon \times (0, T) \end{aligned} \quad (2.6)$$

Initial Condition

$$u^\epsilon(x^\epsilon, 0) = 0, \quad \phi^\epsilon(x^\epsilon, 0) = \bar{\phi}^\epsilon(x^\epsilon) \quad (2.7)$$

The unknowns of the above system are the displacement vector field $u^\epsilon(x^\epsilon, t)$, corresponding to the displacement of the point x^ϵ of the rod at the time t , and $\phi^\epsilon(x^\epsilon, t)$ which stands for the measure of change in volume fraction of the elastic material from a reference volume fraction configuration ξ_0 . (σ_{ij}^ϵ) is the stress tensor and e_{ij}^ϵ is the linear strain tensor.

The data of the system are the following: the density γ of the full elastic material is a constant and independent of ϵ ; the body load $f^\epsilon = (f_i^\epsilon)$ depends only on t ; the normal tractions on the boundary g^ϵ and h^ϵ ; the coefficients $a_{ijkl}^\epsilon(\phi^\epsilon)$, $b_{ijkl}^\epsilon(\phi^\epsilon)$, $a^\epsilon(\phi^\epsilon)$, $A_{ij}^\epsilon(\phi^\epsilon)$, $B_{ijkl}^\epsilon(\phi^\epsilon)$ are all material coefficients depending on the change in volume fraction; η and ρ are small parameters; the truncation operator $P_\eta(\cdot)$ and the mollification operator M_ρ are defined as:

$$P_\eta(\phi)(x) = \begin{cases} -\xi_0(x) + \frac{\eta}{2}, & \text{if } \phi(x) \leq -\xi_0(x) + \frac{\eta}{2} \\ \phi, & \text{if } \eta - \xi_0 \leq \phi \leq 1 - \xi_0 - \eta \\ 1 - \xi_0, & \text{if } \phi \geq 1 - \xi_0 \end{cases} \quad (2.8)$$

and satisfies

$$0 < \frac{\eta}{2} \leq (\xi_0 + P_\eta(\phi)) \leq 1 \quad (2.9)$$

$$M_\rho(g)(x) = \int_{R^3} w_\rho(x - y) \bar{g}(y) dy \quad (2.10)$$

where the function \bar{g} is the extension of g to R^3 ; w_ρ is a mollifier,

$$w_\rho(x) = \left(\int_{R^3} w(y) dy \right)^{-1} \frac{1}{\rho^n} w\left(\frac{x}{\rho}\right), \quad (2.11)$$

where w is the function defined by

$$w(x) = \begin{cases} \exp(1/(|x| - 1)), & \text{if } |x| < 1, \\ 0, & \text{if } |x| > 1 \end{cases} \quad (2.12)$$

The coefficients $a_{ijkl}^\epsilon(\phi^\epsilon)$, $b_{ijkl}^\epsilon(\phi^\epsilon)$ are elasticity and viscosity coefficients, $c_{ijkl}^\epsilon(\phi^\epsilon)$, d_{ijkl}^ϵ are the modified elasticity and viscosity coefficients; $a_{ijkl}^\epsilon(\phi^\epsilon)$, b_{ijkl}^ϵ satisfy the following symmetric and ellipticity conditions

$$\begin{aligned} a_{ijkl}^\epsilon(\phi^\epsilon) &= a_{jikl}^\epsilon(\phi^\epsilon) = a_{klij}^\epsilon(\phi^\epsilon), \\ a_{\alpha\beta\gamma\delta}^\epsilon(\phi^\epsilon) &= a_{\alpha\delta\beta\gamma}^\epsilon(\phi^\epsilon) = 0, \\ (\xi_0 + \phi^\epsilon) a_{ijkl}^\epsilon(\phi^\epsilon) \tau_{ij} \tau_{kl} &\geq c_1 \tau_{ij} \tau_{ij} \end{aligned} \quad (2.13)$$

$$\begin{aligned}
b_{ijkl}^\epsilon(\phi^\epsilon) &= b_{jikl}^\epsilon(\phi^\epsilon) = b_{klij}^\epsilon(\phi^\epsilon), \\
b_{\alpha\beta\gamma 3}^\epsilon(\phi^\epsilon) &= b_{\alpha 333}^\epsilon(\phi^\epsilon) = 0, \\
(\xi_0 + \phi^\epsilon)b_{ijkl}^\epsilon(\phi^\epsilon)\tau_{ij}\tau_{kl} &\geq c_2\tau_{ij}\tau_{ij}
\end{aligned} \tag{2.14}$$

where c_1, c_2 are positive constants independent of ϵ . Obviously, $c_{ijkl}^\epsilon, d_{ijkl}^\epsilon$ satisfy (2.13) and (2.14) respectively.

$A_{ij}^\epsilon(\phi^\epsilon), \bar{A}_{ij}^\epsilon(\phi^\epsilon), \bar{B}_{ijkl}^\epsilon$ and B_{ijkl}^ϵ are the remodelling rate coefficients.

3 Asymptotic Expansion Method

A major geometric feature of a three-dimensional rod is the fact that the measure of the largest cross sectional dimension is very small when compared to its length ($\epsilon \ll L$). This fact has driven mechanists to search for simpler models. Based on a priori hypotheses from experience, lower dimension models were derived such as the classical Bernoulli-Navier equations. One of the purposes in this section is to study the behavior of the solution $(u^\epsilon, \phi^\epsilon)$ as $\epsilon \rightarrow 0$. In order to do so we use an asymptotic expansion method with respect to the small dimension as in the elastic rod case [8]. The dependence of the solution $(u^\epsilon, \phi^\epsilon)$ with respect to ϵ is rather complex. The technique of change of variable to a fixed domain (reference rod) and the subsequent rescaling of the displacement and the change in volume fraction allows us to define an equivalent problem to (2.3)-(2.7) where ϵ shows up in an explicit way in the rescaled equations.

The reference domain is

$$\Omega = \omega \times (0, L) \tag{3.1}$$

and the corresponding boundary subsets are

$$\Gamma = \partial\omega \times (0, L), \Gamma_0 = \bar{\omega} \times \{0\}, \Gamma_L = \bar{\omega} \times \{L\}, \tag{3.2}$$

where $\partial\omega$ is the boundary of ω .

With every point $x \in \bar{\Omega}$ we associate the point $x^\epsilon \in \bar{\Omega}^\epsilon$ through the following transformation:

$$\Pi^\epsilon : x = (x_1, x_2, x_3) \in \bar{\Omega} \rightarrow x^\epsilon = (x_1^\epsilon, x_2^\epsilon, x_3^\epsilon) = (\epsilon x_1, \epsilon x_2, x_3) \in \bar{\Omega}^\epsilon \tag{3.3}$$

Consequently we have

$$\Gamma^\epsilon = \Pi^\epsilon(\Gamma), \quad \Gamma_0^\epsilon = \Pi^\epsilon(\Gamma_0), \quad \Gamma_L^\epsilon = \Pi^\epsilon(\Gamma_L). \tag{3.4}$$

In order to get an equivalent problem to (2.3)-(2.7) in set Ω , we associate with the displacements u^ϵ and with functions v^ϵ the scaled displacements $u(\epsilon)$ and scaled functions v through the following scalings for all $x^\epsilon = \Pi^\epsilon(x), x \in \bar{\Omega}$:

$$\begin{aligned} u_\alpha(\epsilon)(x) &= \epsilon u_\alpha^\epsilon(x^\epsilon), v_\alpha(x) = \epsilon v_\alpha^\epsilon(x^\epsilon) \\ u_3(\epsilon)(x) &= u_3^\epsilon(x^\epsilon), v_3(x) = v_3(x^\epsilon). \end{aligned} \quad (3.5)$$

We associate with the change in volume fraction ϕ^ϵ the scaled change in volume fraction $\phi(\epsilon)$ through

$$\phi(\epsilon)(x, t) = \phi^\epsilon(x^\epsilon, t) \quad (3.6)$$

We assume the data satisfy

$$\begin{aligned} f_\alpha^\epsilon(x^\epsilon, t) &= \epsilon f_\alpha(x, t), g_\alpha^\epsilon(x^\epsilon, t) = \epsilon^2 g_\alpha(x, t), h_\alpha^\epsilon(x^\epsilon, t) = \epsilon h_\alpha(x, t), \\ f_3^\epsilon(x^\epsilon, t) &= f_3(x, t), g_3^\epsilon(x^\epsilon, t) = \epsilon g_3(x, t), h_3^\epsilon(x^\epsilon, t) = h_3(x, t), \end{aligned} \quad (3.7)$$

where f_i, g_i and h_i are independent of ϵ .

We suppose that the scaled displacement $u(\epsilon)$ has a formal asymptotic expansion:

$$u(\epsilon) = u^0 + \epsilon u^1 + \epsilon^2 u^2 + \dots \quad (3.8)$$

where u^i depend on x, t and are independent of ϵ . We also assume

$$\phi(\epsilon) = \phi^0 + \epsilon \phi^1 + \epsilon^2 \phi^2 + \dots, \quad (3.9)$$

We also assume that

$$\begin{aligned} P_\eta(\phi^\epsilon)(x^\epsilon, t) &= P_\eta(\phi^0)(x, t) \\ a_{ijkl}^\epsilon(\phi^\epsilon)(x^\epsilon, t) &= a_{ijkl}(\phi_0)(x, t), c_{ijkl}^\epsilon(\phi^\epsilon)(x^\epsilon, t) = c_{ijkl}(\phi_0)(x, t), \\ b_{ijkl}^\epsilon(\phi^\epsilon)(x^\epsilon, t) &= b_{ijkl}(\phi_0)(x, t), d_{ijkl}^\epsilon(\phi^\epsilon)(x^\epsilon, t) = d_{ijkl}(\phi_0)(x, t), \\ A_{ij}^\epsilon(\phi^\epsilon)(x^\epsilon, t) &= A_{ij}(x, t), a^\epsilon(\phi^\epsilon)(x^\epsilon, t) = a(\phi^0)(x, t), B_{ijkl}^\epsilon(\phi^\epsilon) = B_{ijkl}(\phi_0) \end{aligned} \quad (3.10)$$

where $P_\eta, a_{ijkl}, b_{ijkl}, c_{ijkl}, d_{ijkl}, a, B_{ijkl}$ and A_{ij} are independent of ϵ . Using these scalings and hypotheses, we reformulate the rod problem in the fixed reference domain:

Theorem 3.1. *The scaled displacements $u(\epsilon)$ and the scaled change in volume fraction $d(\epsilon)$ satisfy the following equations:*

$$\begin{aligned} -\partial_j \sigma_{ij}(\epsilon) &= \gamma(\xi_0 + P_\eta(\phi_0)) f_i, \text{ in } \Omega \times (0, T), \\ \sigma_{ij}(\epsilon) n_j &= g_i, \text{ in } \Gamma \times (0, T), \\ \sigma_{ij}(\epsilon) n_j &= h_i, \text{ in } (\Gamma_0 \cup \Gamma_L) \times (0, T) \end{aligned} \quad (3.11)$$

$$\begin{aligned}
\dot{\phi}(\epsilon) = & \epsilon^{-4} B_{\alpha\beta\gamma\mu}(\phi_0) e_{\alpha\beta}(\epsilon) e_{\mu\gamma}(\epsilon) + \\
& + \epsilon^{-2} (A_{\alpha\beta}(\phi^0) e_{\alpha\beta}(\epsilon) + \bar{A}_{\alpha\beta}(\phi^0) \dot{e}_{\alpha\beta}(\epsilon) + 2B_{\alpha\beta 33}(\phi^0) e_{\alpha\beta}(\epsilon) e_{33}(\epsilon) \\
& + 2\bar{B}_{\alpha\beta 33}(\phi^0) \dot{e}_{\alpha\beta}(\epsilon) \dot{e}_{33}(\epsilon) + 4B_{3\alpha\beta 3}(\phi^0) e_{3\alpha}(\epsilon) e_{3\beta}(\epsilon) + 4\bar{B}_{3\alpha\beta 3}(\phi^0) \dot{e}_{3\alpha}(\epsilon) \dot{e}_{3\beta}(\epsilon)) + \\
& + \epsilon^{-1} [(A_{3\alpha}(\phi^0) + A_{\alpha 3}(\phi^0)) e_{3\alpha}(\epsilon) + (\bar{A}_{3\alpha}(\phi^0) + \bar{A}_{\alpha 3}(\phi^0)) \dot{e}_{3\alpha}(\epsilon)] \\
& + a(\phi^0) + A_{33}(\phi^0) e_{33}(\epsilon) + \bar{A}_{33}(\phi^0) \dot{e}_{33}(\epsilon) + B_{3333}(\phi^0) e_{33}(\epsilon) e_{33}(\epsilon) + \bar{B}_{3333}(\phi^0) \dot{e}_{33}(\epsilon) e_{33}(\epsilon), \text{ in } \bar{\Omega} \times \mathbb{R}^+
\end{aligned} \tag{3.12}$$

$$u^\epsilon(x, 0) = 0, \quad \phi(\epsilon)(x, 0) = \bar{\phi}(\epsilon)(x), \text{ in } \bar{\Omega} \tag{3.13}$$

where $\bar{\phi}(\epsilon)$ is the scaled initial condition.

Proof. The proof follows as in [8], if one takes into account the following relationship between the linearized strain tensor $e^\epsilon(u^\epsilon)$ and the scaled linearized strain tensor $e(u(\epsilon))$:

$$e_{\alpha\beta}(u(\epsilon))(x) = \epsilon^2 e_{\alpha\beta}^\epsilon(u^\epsilon)(x^\epsilon), \quad e_{3\beta}(u(\epsilon))(x) = \epsilon e_{3\beta}^\epsilon(u^\epsilon)(x^\epsilon), \quad e_{33}(u(\epsilon))(x) = e_{33}^\epsilon(u^\epsilon)(x^\epsilon). \tag{3.14}$$

and it follows that

$$\sigma_{\alpha\beta}(\epsilon)(x) = \epsilon^{-2} \sigma_{\alpha\beta}^\epsilon(x^\epsilon), \quad \sigma_{3\beta}(\epsilon)(x) = \epsilon^{-1} \sigma_{3\beta}^\epsilon(x^\epsilon), \quad \sigma_{33}(\epsilon)(x) = \sigma_{33}^\epsilon(x^\epsilon) \tag{3.15}$$

□

According to [8], it can be proved that the asymptotic expansion (3.8) does not contain odd powers of ϵ , that is

$$u(\epsilon) = u^0 + \epsilon^2 u^2 + \epsilon^4 u^4 + \dots \tag{3.16}$$

This gives the following expansion on the scaled strain tensor

$$e_{ij}(u(\epsilon)) = e_{ij}(u^0) + \epsilon^2 e_{ij}(u^2) + \epsilon^4 e_{ij}(u^4) + \dots = e_{ij}^0 + \epsilon^2 e_{ij}^2 + \epsilon^4 e_{ij}^4 + \dots \tag{3.17}$$

Consequently, we have

$$\begin{aligned}
\sigma_{\alpha\beta}(\epsilon) &= \epsilon^{-4} \sigma_{\alpha\beta}^{-4} + \epsilon^{-2} \sigma_{\alpha\beta}^{-2} + \sigma_{\alpha\beta}^0 + \dots \\
\sigma_{3\beta}(\epsilon) &= \epsilon^{-2} \sigma_{3\beta}^{-2} + \sigma_{3\beta}^0 + \epsilon^2 \sigma_{3\beta}^2 + \dots \\
\sigma_{33}(\epsilon) &= \epsilon^{-2} \sigma_{33}^{-2} + \sigma_{33}^0 + \epsilon^2 \sigma_{33}^2 + \dots
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
\sigma_{\alpha\beta}^{-4} &= c_{\alpha\beta\gamma\mu}e_{\gamma\mu}(u^0) + d_{\alpha\beta\gamma\mu}e_{\gamma\mu}(\dot{u}^0) \\
\sigma_{\alpha\beta}^p &= c_{\alpha\beta\gamma\mu}e_{\gamma\mu}(u^{p+4}) + c_{\alpha\beta 33}e_{33}(u^{p+2}) + d_{\alpha\beta\gamma\mu}e_{\gamma\mu}(\dot{u}^{p+4}) + d_{\alpha\beta 33}e_{33}(\dot{u}^{p+2}) \text{ for } p \geq -2 \\
\sigma_{3\beta}^p &= 2c_{3\beta 3\mu}e_{3\mu}(u^{p+2}) + 2d_{3\beta 3\mu}e_{3\mu}(\dot{u}^{p+2}) \\
\sigma_{33}^p &= c_{3333}e_{33}(u^p) + c_{\alpha\beta 33}e_{\alpha\beta}(u^{p+2}) + d_{3333}e_{33}(\dot{u}^p) + d_{\alpha\beta 33}e_{\alpha\beta}(\dot{u}^{p+2}) \text{ for } p \geq 0
\end{aligned} \tag{3.19}$$

Using the same techniques as in [8], we can prove that

$$\sigma_{ij}^p = 0, \text{ for } p < 0 \tag{3.20}$$

so the asymptotic expansion for the scaled stress tensor is

$$\sigma(\epsilon) = \sigma^0 + \epsilon^2\sigma^2 + \epsilon^4\sigma^4 + \dots \tag{3.21}$$

We introduce the coefficients $D_{ijkl}(\phi^0)$ of the inverse matrix defined by the viscoelastic coefficients $d_{ijkl}(\phi^0)$, that is

$$\tau_{ij} = D_{ijkl}\bar{\tau}_{kl}, \quad \bar{\tau}_{ij} = d_{ijkl}\tau_{kl} \tag{3.22}$$

for any tensors τ and $\bar{\tau}$.

From (2.4), we have

$$\frac{d e_{kl}}{dt} + D_{ijkl}(\phi^0)c_{ijmp}(\phi^0)e_{mp} = D_{ijkl}(\phi^0)\sigma_{ij} \tag{3.23}$$

In the above system of ordinary differential equations, solving for e_{kl} , we obtain

$$e_{kl}(t) = e_{kj}(0) + \int_0^t \exp\left(\int_t^s D_{ijkl}c_{ijmp}d\tau\right) D_{ijmp}\sigma_{mp}ds \tag{3.24}$$

since we have $e_{kl}(0) = 0$, we can define $T_{klnq}(t)$ so that

$$T_{klnq}(t, s) = \left[e^{\int_t^s d_{ijkl}(\tau)c_{ijmp}(\tau) d\tau} \right] D_{nqmp}(s) \tag{3.25}$$

Then we have

$$e_{kl}(t) = \int_0^t T_{klij}(t, s)\sigma_{ij}(s) \tag{3.26}$$

We introduce the following functions of x_3

$$l(x_3, t) = \int_{\omega} \frac{1}{T_{3333}}d\omega, e_{\alpha}(x_3, t) = \int_{\omega} \frac{x_{\alpha}}{T_{3333}}d\omega, h_{\alpha\beta}(x_3, t) = \int_{\omega} \frac{x_{\alpha}x_{\beta}}{T_{3333}}d\omega, \tag{3.27}$$

We have the following theorem

Theorem 3.2. *The first term $u^0 = (u_1^0, u_2^0, u_3^0)$ in the asymptotic expansion (3.16) satisfies*

$$u_\alpha^0 = u_\alpha^0(x_3, t), u_3^0 = \bar{u}_3^0(x_3, t) - x_\alpha \partial_3 u_\alpha^0 \quad (3.28)$$

where u_α^0 and \bar{u}_3^0 depend only on x_3 and t , and satisfy the following generalized evolutionary Bernoulli-Navier model

$$\begin{aligned} -\partial_3[l\partial_t(\partial_3\bar{u}_3^0 - e_\alpha\partial_{33}u_\alpha^0)] &= \int_\omega \gamma(\xi_0 + P_\eta(\phi^0))f_3d\omega + \int_{\partial\omega} g_3d\partial\omega \\ \partial_{33t}(-e_\beta\partial_3\bar{u}_3^0 + h_{\alpha\beta}\partial_{33}u_\alpha^0) &= \int_\omega \gamma(\xi_0 + P_\eta(\phi^0))f_\beta d\omega + \int_{\partial\omega} g_\beta d\partial\omega \\ &+ \int_\omega x_\beta\partial_3[\gamma(\xi_0 + P_\eta(\phi^0))f_3]d\omega + \int_{\partial\omega} x_\beta\partial_3g_3d\partial\omega \end{aligned} \quad (3.29)$$

Boundary conditions for $\{\bar{x}_3\} \times (0, T)$, with $\bar{x}_3 = 0, L$

$$\begin{aligned} (\partial_t l \partial_3 \bar{u}_3^0 - e_\alpha \partial_{33} u_\alpha^0)(\bar{x}_3) &= \int_\omega h_3(\bar{x}_3) d\omega \\ \partial_t (e_\beta \partial_3 \bar{u}_3^0 - h_{\alpha\beta} \partial_{33} u_\alpha^0)(\bar{x}_3) &= \int_\omega x_\beta h_3(\bar{x}_3) d\omega \\ \partial_t \partial_3 (e_\beta \partial_3 \bar{u}_3^0 - h_{\alpha\beta} \partial_{33} u_\alpha^0)(\bar{x}_3) &= \int_\omega h_\beta(\bar{x}_3) d\omega - \int_\omega x_\beta \gamma(\xi_0 + P_\eta(\phi^0)) f_3 d\omega - \int_{\partial\omega} x_\beta \partial_3 g_3 d\partial\omega \end{aligned} \quad (3.30)$$

and initial condition

$$u^0|_{t=0} = 0 \quad (3.31)$$

Moreover the term σ_{33}^0 satisfies

$$\sigma_{33}^0 = \frac{1}{T_{3333}(\phi^0)} \partial_t (\partial_3 \bar{u}_3^0 - x_\alpha \partial_{33} u_\alpha^0) = \frac{1}{T_{3333}(\phi^0)} \partial_t (\partial_3 u_3^0) = \frac{1}{T_{3333}} e_{33} (\partial_t u^0) \quad (3.32)$$

Proof. We are going to use the mixed displacement-stress approach as in [8]. Using equation (2.3) and (3.26), we have

$$\int_{\Omega^\epsilon} \int_0^t T_{klnq} \sigma_{nq} ds \tau_k l^\epsilon dx^\epsilon - \int_{\Omega^\epsilon} e_{ij}^\epsilon(u^\epsilon) \tau_{kl} dx^\epsilon = 0 \quad (3.33)$$

$$\int_{\Omega^\epsilon} \sigma_{ij}^\epsilon e_{ij}^\epsilon(v^\epsilon) dx^\epsilon = \int_{\Omega^\epsilon} \gamma(\xi_0 + P_\eta(\phi_0)) f_i^\epsilon v_i^\epsilon + \int_{\Gamma_\epsilon} g_i^\epsilon v_i^\epsilon + \int_{\Gamma_\epsilon} h_i^\epsilon v_i^\epsilon \quad (3.34)$$

where τ_{kl} and v_i^ϵ are test functions.

Introducing the same change of variable and scalings of (3.3) and (3.5) we have

$$a_0(\sigma(\epsilon), \tau) + \epsilon^2 a_2(\sigma(\epsilon), \tau) + \epsilon^4 a_4(\sigma(\epsilon), \tau) + b(\tau, u(\epsilon)) = 0 \quad (3.35)$$

for all test tensor τ .

$$b(\sigma(\epsilon), v) = \int_{\Omega} \gamma(\xi_0 + P_{\eta}(\phi_0)) f_i v_i + \int_{\Gamma} g_i v_i + \int_{\Gamma} h_i v_i \quad (3.36)$$

for all test functions v .

where the following bilinear forms are defined:

$$a_0(\sigma, \tau) = \int_{\Omega} \left(\int_0^t T_{3333} \sigma_{33} ds \right) \tau_{33} dx \quad (3.37)$$

$$a_2(\sigma, \tau) = \int_{\Omega} \left[4 \left(\int_0^t T_{3\alpha 3\beta} \sigma_{3\alpha} \right) \tau_{3\beta} + \left(\int_0^t T_{\alpha\beta 33} (\sigma_{33} \tau_{\alpha\beta} + \sigma_{\alpha\beta} \tau_{33}) \right) \right] dx \quad (3.38)$$

$$a_4(\sigma, \tau) = \int_{\Omega} \left(\int_0^t T_{\alpha\beta nq} \sigma_{nq} \right) \tau_{\alpha\beta} \quad (3.39)$$

$$b(\tau, v) = - \int_{\Omega} \tau_{ij} e_{ij}(v) dx \quad (3.40)$$

From (3.37) - (3.40) and , we obtain the following:

$$\int_{\Omega} \left(\int_0^t T_{3333} \sigma_{33} \right) \tau_{33} dx - \int_{\Omega} \partial_3 u_3^0 \tau_{33} dx = 0 \quad (3.41)$$

$$\int_{\Omega} (\partial_3 u_{\beta}^0 + \partial_{\beta} u_3^0) \tau_{3\beta} dx = 0 \quad (3.42)$$

$$-\frac{1}{2} \int_{\Omega} (\partial_{\alpha} u_{\beta}^0 + \partial_{\beta} u_{\alpha}^0) \tau_{\alpha\beta} dx = 0 \quad (3.43)$$

$$\int_{\Omega} (\sigma_{33}^0 \partial_3 v_3 + \sigma_{3\beta}^0 \partial_{\beta} v_3) dx = \int_{\Omega} \gamma(\xi_0 + P_{\eta}(\phi_0)) f_3 v_3 + \int_{\Gamma} g_3 v_3 + \int_{\Gamma} h_3 v_3 \quad (3.44)$$

$$\int_{\Omega} (\sigma_{\alpha\beta}^0 \partial_{\alpha} v_{\beta} + \sigma_{3\beta}^0 \partial_3 v_{\beta}) dx = \int_{\Omega} \gamma(\xi_0 + P_{\eta}(\phi_0)) f_{\beta} v_{\beta} + \int_{\Gamma} g_{\beta} v_{\beta} + \int_{\Gamma} h_{\beta} v_{\beta} \quad (3.45)$$

$$\int_{\Omega} \left(\int_0^t T_{3333} \sigma_{33}^2 ds \right) \tau_{33} dx - \int_{\Omega} \partial_3 u_3^2 \tau_{33} dx = - \int_{\Omega} \left(\int_0^t T_{\alpha\beta 33} \sigma_{\alpha\beta}^2 ds \right) \tau_{33} dx \quad (3.46)$$

$$\int_{\Omega} (\partial_3 u_{\beta}^2 + \partial_{\beta} u_3^2) \tau_{3\beta} dx = \int_{\Omega} 4 \left(\int_0^t T_{3\alpha 3\beta} \sigma_{3\alpha}^2 ds \right) \tau_{3\beta} dx \quad (3.47)$$

$$\frac{1}{2} \int_{\Omega} (\partial_{\alpha} u_{\beta}^2 + \partial_{\beta} u_{\alpha}^2) \tau_{\alpha\beta} dx = \int_{\Omega} \left(\int_0^t T_{\alpha\beta 33} \sigma_{33}^0 ds \right) \tau_{\alpha\beta} dx \quad (3.48)$$

$$\int_{\Omega} (\sigma_{33}^2 \partial_3 v_3 + \sigma_{3\beta}^2 \partial_{\beta} v_3) dx = 0 \quad (3.49)$$

$$\int_{\Omega} (\sigma_{\alpha\beta}^2 \partial_{\alpha} v_{\beta} + \sigma_{3\beta}^2 \partial_3 v_{\beta}) dx = 0 \quad (3.50)$$

Now (3.42) and (3.43) warrant (3.28); (3.32) is a direct consequence of equation (3.41). The first equations of (3.29) and (3.30) follow from (3.44) and (3.32). To obtain the second one, plugging $v_3 = x_\beta \partial_3 v_\beta$ into (3.44), $v_\beta = \partial v_\beta^0$ into (3.45), we obtain, after subtraction, that

$$-\int_0^L \left[\int_\omega x_\beta \sigma_3^0 \mathfrak{z} \right] \partial_3 \mathfrak{z} v_\beta dx_3 = \int_\Omega \gamma(\xi_0 + P_\eta(\phi_0)) f_3 v_3 + \int_\Gamma g_3 v_3 + \int_\Gamma h_3 v_3 \quad (3.51)$$

The above equation and (3.28) implies the second equation of (3.29) and the second and third equations of (3.30)

□

Now we can identify and derive some conclusions about the terms in the expansion of $\phi(\epsilon)$. Assume that the scaled initial condition $\bar{\phi}(\epsilon)$ is of the following form

$$\bar{\phi}(\epsilon) = \bar{\phi}^0 + \bar{\phi}^1 \epsilon + \bar{\phi}^2 \epsilon^2 + \dots \quad (3.52)$$

where $\bar{\phi}^k, k = 0, 1, 2, \dots$ are functions defined in $\bar{\Omega}$ but independent of ϵ .

Theorem 3.3. *The terms ϕ^0, ϕ^1 and ϕ^2 of the asymptotic expansion satisfy the following ordinary differential equations:*

$$\begin{aligned} \dot{\phi}^0 = & B_{\alpha\beta\gamma\mu} \mathcal{E}_{\alpha\beta}^0 \mathcal{E}_{\gamma\mu}^0 + \bar{B}_{\alpha\beta\gamma\mu} \dot{\mathcal{E}}_{\alpha\beta}^0 \dot{\mathcal{E}}_{\gamma\mu}^0 + A_{\alpha\beta} \mathcal{E}_{\alpha\beta}^0 + \bar{A}_{\alpha\beta} \dot{\mathcal{E}}_{\alpha\beta}^0 \\ & + 2B_{\alpha\beta 33} \mathcal{E}_{\alpha\beta}^0 e_{33}^2 + 2\bar{B}_{\alpha\beta 33} \dot{\mathcal{E}}_{\alpha\beta}^0 \dot{\mathcal{E}}_{33}^0 + A_{33} e_{33}^0 + \bar{A}_{33} \dot{e}_{33}^0 + B_{3333} e_{33}^0 e_{33}^0 + \bar{B}_{3333} \dot{e}_{33}^0 \dot{e}_{33}^0 + a \end{aligned} \quad (3.53)$$

$$\phi^0(0) = \bar{\phi}^0 \quad (3.54)$$

where \mathcal{E}^0 is defined by

$$\mathcal{E}_{\alpha\beta}^0 = \int_0^t T_{33\alpha\beta}(t, s) \left[-c_{3333} e_{33}^0(s) - \left(d_{3333} - \frac{1}{T_{3333}} \right) \dot{e}_{33}^0(s) \right] ds \quad (3.55)$$

Proof. The initial condition follows from (3.13) and (3.52).

Plugging (3.17) into (3.12) we have

$$\begin{aligned}
\epsilon^4 \dot{\phi}^0 + \epsilon^5 \dot{\phi}^1 + \epsilon^6 \dot{\phi}^2 + \dots = & B_{\alpha\beta\gamma\mu}(e_{\alpha\beta}^0 + \epsilon^2 e_{\alpha\beta}^2 + \epsilon^4 e_{\alpha\beta}^4 + \dots)(e_{\gamma\mu}^0 + \epsilon^2 e_{\gamma\mu}^2 + \epsilon^4 e_{\gamma\mu}^4 + \dots) \\
& + \bar{B}_{\alpha\beta\gamma\mu}(\dot{e}_{\alpha\beta}^0 + \epsilon^2 \dot{e}_{\alpha\beta}^2 + \epsilon^4 \dot{e}_{\alpha\beta}^4 + \dots)(\dot{e}_{\gamma\mu}^0 + \epsilon^2 \dot{e}_{\gamma\mu}^2 + \epsilon^4 \dot{e}_{\gamma\mu}^4 + \dots) \\
& + \epsilon^2 [A_{\alpha\beta}(e_{\alpha\beta}^0 + \epsilon^2 e_{\alpha\beta}^2 + \epsilon^4 e_{\alpha\beta}^4 + \dots) + \bar{A}_{\alpha\beta}(\dot{e}_{\alpha\beta}^0 + \epsilon^2 \dot{e}_{\alpha\beta}^2 + \epsilon^4 \dot{e}_{\alpha\beta}^4 + \dots) \\
& + 2B_{\alpha\beta 33}(e_{\alpha\beta}^0 + \epsilon^2 e_{\alpha\beta}^2 + \epsilon^4 e_{\alpha\beta}^4 + \dots)(e_{33}^0 + \epsilon^2 e_{33}^2 + \epsilon^4 e_{33}^4 + \dots) \\
& + 2\bar{B}_{\alpha\beta 33}(\dot{e}_{\alpha\beta}^0 + \epsilon^2 \dot{e}_{\alpha\beta}^2 + \epsilon^4 \dot{e}_{\alpha\beta}^4 + \dots)(\dot{e}_{33}^0 + \epsilon^2 \dot{e}_{33}^2 + \epsilon^4 \dot{e}_{33}^4 + \dots) \\
& + 4B_{3\alpha 3\beta}(e_{3\alpha}^0 + \epsilon^2 e_{3\alpha}^2 + \epsilon^4 e_{3\alpha}^4 + \dots)(e_{3\beta}^0 + \epsilon^2 e_{3\beta}^2 + \epsilon^4 e_{3\beta}^4 + \dots) \\
& + 4\bar{B}_{3\alpha 3\beta}(\dot{e}_{3\alpha}^0 + \epsilon^2 \dot{e}_{3\alpha}^2 + \epsilon^4 \dot{e}_{3\alpha}^4 + \dots)(\dot{e}_{3\beta}^0 + \epsilon^2 \dot{e}_{3\beta}^2 + \epsilon^4 \dot{e}_{3\beta}^4 + \dots)] \\
& + \epsilon^3 [(A_{3\alpha} + A_{\alpha 3})(e_{3\alpha}^0 + \epsilon^2 e_{3\alpha}^2 + \epsilon^4 e_{3\alpha}^4 + \dots) + (\bar{A}_{3\alpha} + \bar{A}_{\alpha 3})(\dot{e}_{3\alpha}^0 + \epsilon^2 \dot{e}_{3\alpha}^2 + \epsilon^4 \dot{e}_{3\alpha}^4 + \dots)] \\
& + \epsilon^4 [a + A_{33}(e_{33}^0 + \epsilon^2 e_{33}^2 + \epsilon^4 e_{33}^4 + \dots) + \bar{A}_{33}(\dot{e}_{33}^0 + \epsilon^2 \dot{e}_{33}^2 + \epsilon^4 \dot{e}_{33}^4 + \dots) + \\
& + B_{3333}(e_{33}^0 + \epsilon^2 e_{33}^2 + \epsilon^4 e_{33}^4 + \dots)(e_{33}^0 + \epsilon^2 e_{33}^2 + \epsilon^4 e_{33}^4 + \dots) \\
& + \bar{B}_{3333}(\dot{e}_{33}^0 + \epsilon^2 \dot{e}_{33}^2 + \epsilon^4 \dot{e}_{33}^4 + \dots)(\dot{e}_{33}^0 + \epsilon^2 \dot{e}_{33}^2 + \epsilon^4 \dot{e}_{33}^4 + \dots)]
\end{aligned} \tag{3.56}$$

From Theorem 3.2, we have

$$e_{\alpha\beta}^0 = e_{\alpha\beta}(u^0) = 0, \quad e_{3\alpha}^0 = e_{3\alpha}(u^0) = 0, \tag{3.57}$$

Using (3.57), we obtain that the coefficients of ϵ^0 on the right hand side of (3.56) are equal to 0, so are the coefficients of ϵ^2 on the right hand side of (3.56).

For the coefficients of ϵ^4 we obtain

$$\begin{aligned}
\dot{\phi}^0 = & B_{\alpha\beta\gamma\mu} e_{\alpha\beta}^2 e_{\gamma\mu}^2 + \bar{B}_{\alpha\beta\gamma\mu} \dot{e}_{\alpha\beta}^2 \dot{e}_{\gamma\mu}^2 + A_{\alpha\beta} e_{\alpha\beta}^2 + \bar{A}_{\alpha\beta} \dot{e}_{\alpha\beta}^2 \\
& + 2B_{\alpha\beta 33} e_{\alpha\beta}^2 e_{33}^2 + 2\bar{B}_{\alpha\beta 33} \dot{e}_{\alpha\beta}^2 \dot{e}_{33}^2 + A_{33} e_{33}^0 + \bar{A}_{33} \dot{e}_{33}^0 + B_{3333} e_{33}^0 e_{33}^0 + \bar{B}_{3333} \dot{e}_{33}^0 \dot{e}_{33}^0 + a
\end{aligned} \tag{3.58}$$

Now using the fourth equation of (3.19) with $p = 0$, we have

$$d_{\alpha\beta 33} \dot{e}_{\alpha\beta}^2 + c_{\alpha\beta 33} e_{\alpha\beta}^2 = \sigma_{33}^0 - c_{3333} e_{33}^0 - d_{3333} \dot{e}_{33}^0 \tag{3.59}$$

From (3.24) and (3.26), we obtain

$$e_{\alpha\beta}^2 = \int_0^t T_{33\alpha\beta}(t, s) [\sigma_{33}^0(s) - c_{3333} e_{33}^0(s) - d_{3333} \dot{e}_{33}^0(s)] ds \tag{3.60}$$

Using (3.60), (3.32) and (3.58), we obtain (3.53) and (3.55).

□

References

- [1] P.G. Ciarlet, *Mathematical Elasticity, Vol. 1: Three-Dimensional Elasticity*, Studies in Mathematics and its Applications, Vol. 20, North-Holland, Amsterdam, 1988.
- [2] S.C. Cowin, D.H.Hegedus, Bone remodelling I: theory of adaptive elasticity, *Journal of Elasticity*, **6**(3), 313-326, 1976.
- [3] S.C. Cowin, D.H.Hegedus, Bone remodelling II: small strain adaptive elasticity, *Journal of Elasticity*, **6**(4), 337-352, 1976.
- [4] S.C. Cowin, R.R. Nachlinger, , Bone remodelling III: Uniqueness and stability in adaptive elasticity, *Journal of Elasticity*, **8**(3), 285-295, 1978.
- [5] I. Figueiredo, L. Trabucho, Asymptotic model of a nonlinear adaptive elastic rod, *Math. Mech. Solids* **no. 4**, 331-353, 2004.
- [6] W. Maurel, Y. Wu, N. Magnenat Thalmann, D. Thalmann, *Biomedical Models for Soft Tissue*, Springer (1998) Berlin.
- [7] J. Monnier, L. Trabucho, An existence and uniqueness result in bone remodeling theory, *Computer Meth.. Applied Mech. Eng.*, 151, 539-544, 1998.
- [8] L. Trabucho, J.M. Viano, Mathematical modelling of rods, in *Handbook of numerical analysis*, p487-974, ed., P.G. Ciale, J.L.Lions, North-Holland, Amsterdam, 1996.
- [9] L. Trabucho, Non-linear bone remodeling: an existence and uniqueness result, *Math. Meth. Appl. Sci.* **23**, 1331-1346 (2000).
- [10] T. Valent, *Boundary value problems of finite elasticity*, Springer Tracts in Natural Philosophy, Vol 31, Springer-Verlag, New York, 1988.