Numerical Homogenization of the Time – Harmonic Acoustics of Bone: The Monophasic Case

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ABSTRACT

In the predecessee to this work, we undertook a derivation of the time-harmonic, acoustic equations, idealizing the bone as a periodic arrangement of a Kelvin-Voigt viscoelastic porous matrix containing a viscous fluid, where we assumed that the fluid was slightly compressible. The effective equations for the monophasic vibrations were obtained, and existence and uniqueness was proved. In this current article, we perform numerical experiments, assuming that the trabeculae are isotropic.

KEYWORDS

two scale convergence, time harmonic wave, viscoelasticity of Kelvin-Voigt

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1. INTRODUCTION

In [1] we developed a theory for the harmonic excitation of cancellous bone where the bone fluid and the elastic matrix move together: the monophasic case.

Using the method of homogenization, we described the microstructure of the composite material, bone plus blood marrow, in terms of a cell problem, where all ingredients exist in equilibrium. The two phases of material are assumed to have the following constitutive equations:

\[
\sigma^t = \theta^t e^{\text{elas}} + (1 - \theta^t) e^{\text{visc}} \tag{1.1}
\]

The viscoelastic behavior of the trabeculae is modeled by a Kelvin-Voigt constitutive equation

\[
\sigma^t_{ij} = (A^t + i\omega B^t)\varepsilon^{\text{elas}}_{ij} \tag{1.2}
\]

Here \(u\) is the wave frequency and \(e(u')\) is the strain tensor defined by

\[
e(u')_{ij} = \frac{1}{2}(\partial_i u'_{j} + \partial_j u'_{i}) \quad i, j = 1, 2, 3
\]

The constants \(A^t_{ij} \) and \( B^t_{ij} \) are the elasticity coefficients of the solid and are assumed to have the classical symmetry and positivity conditions. The constants \( B^t_{ij} \) describe viscosity of the solid, with the classical symmetry and positivity conditions.

The marrow was modeled as a slightly compressible viscous barotropic fluid with the constitutive equations

\[
\sigma^f_{ij} = (A^f + i\omega B^f)\varepsilon^{\text{elas}}_{ij} \tag{1.3}
\]

In (1.3),

\[
A^f_{ij} = c^2 \rho_f b_{ij}
\]

\[
B^f_{ij} = 2\eta_f b_{ij} + \nu b_{ij} \tag{1.4}
\]

Here, \( c \) is the sound speed, \( \rho_f > 0 \) is a constant density of the marrow at rest, and \( \eta_f, \nu \) are constant viscosities, which are subject to the following conditions:

\[\eta_f > 0, \quad \frac{\eta_f}{\nu} > \frac{2}{3}\]

whose physical justification is to be found in [2].

From (1.4), one can obtain more explicit constitutive equations:

\[
u^t := \rho^t \partial_j \nabla \cdot u^t + 2\mu^t \partial^t e(u')^t + \mu^t \nabla \cdot \nabla u^t \tag{1.5}
\]

The equations of motion for the trabeculae (solid part) are given by

\[-\omega^2 \rho_f u^t - \nu \partial_t (\sigma^f_{ij}) = F_{\rho_f} \quad \text{in} \quad \Omega^t \quad \tag{1.6}\]

Here the trabecular stress is defined as (1.2), and \( \rho_f > 0 \) is the constant density of the trabeculae at rest.

In the marrow part,

\[-\omega^2 \rho_f u^t - \nu \partial_t (\sigma^f_{ij}) = F_{\rho_f} \quad \text{in} \quad \Omega^f \quad \tag{1.7}\]

The transition conditions between fluid and solid parts are given by the continuity of displacement

\[u^t = \mathcal{J} u^f \quad \text{on} \quad \Gamma^t \quad \tag{1.8}\]

where \( \mathcal{J} \) indicates the jump across the boundary of \( \Gamma^t = \partial \Omega^t \cap \partial \Omega^f \), and the continuity of the traction

\[\sigma^t_{ij} \cdot n = \sigma^f_{ij} \cdot n \quad \text{on} \quad \Gamma^t \quad \tag{1.9}\]

At the exterior boundary, we imposed the zero Dirichlet condition

\[u^t = 0 \quad \text{on} \quad \partial \Omega^t \quad \tag{1.10}\]

This led to a weak formulation of the slightly compressible problem as

\[-\omega^2 \int_\Omega^t \rho^t u^t (e\hat{\phi}) + \int_\Omega^t \theta^t (A^t + i\omega B^t) e(u')^t : e(\hat{\phi}) + \int_\Omega^t (1 - \theta^t) (A^f + i\omega B^f) e(u')^f : e(\hat{\phi}) = \int_\Omega \rho_f \hat{\phi} \quad \forall \phi \in H^1(\Omega)^n \quad \tag{1.11}\]

where an overbar denotes the complex conjugate.

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2. TWO-SCALE CONVERGENCE

The main convergence results of [1] were obtained by using the method of two-scale convergence. We pass to the limit in the two-scale sense using the weak formulation (1.11), first using a test function \( \phi(x) \), and then a test function \( \frac{1}{\epsilon} \phi \left( \frac{x}{\epsilon} \right) \), where \( \phi(x) \) was assumed to be \( \mathcal{Y} \)-periodic with zero average. This yielded the following equations for \( u^0, u^1 \), the first two terms in the asymptotic expansion of the displacement:

\[
\begin{align*}
\mathcal{A}^0 \cdot \phi = -\omega^2 \mathcal{P}^0 \cdot \phi \\
\mathcal{B}^0 \cdot \phi &= \mathcal{A}^0 \cdot \phi \\
\mathcal{B}^0 \cdot \phi &= \mathcal{A}^0 \cdot \phi
\end{align*}
\]

\( \forall \phi \in H^1(\Omega)^m \)

(2.1)

and

\[
\begin{align*}
0 &= \frac{1}{\epsilon} \left[ \mathcal{A}^1 \cdot \phi + \mathcal{B}^1 \cdot \phi \right] (e_\epsilon (u^0) + e_\epsilon (u^1) : e_\epsilon (\phi)) \\
&\quad + \frac{1}{\epsilon} \left[ (1 - \theta) \mathcal{A}^1 \cdot \phi + \mathcal{B}^1 \cdot \phi \right] (e_\epsilon (u^0) + e_\epsilon (u^1) : e_\epsilon (\phi)) \\
&\quad + \frac{1}{\epsilon} \left[ \mathcal{A}^1 \cdot \phi + \mathcal{B}^1 \cdot \phi \right] (e_\epsilon (u^0) + e_\epsilon (u^1) : e_\epsilon (\phi)) \\
&\quad + \mathcal{C} (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0) (e_\epsilon (u^0) + e_\epsilon (u^1) : e_\epsilon (\phi)) \\
&\quad + \mathcal{C} (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0) (e_\epsilon (u^0) + e_\epsilon (u^1) : e_\epsilon (\phi)) \\
&\quad \forall \phi \in H^1(\Omega)^m / C
\end{align*}
\]

(2.2)

In [1] we constructed the cell problem by substitution of \( u^1 \) given in the special form

\[
u^1(x, y) = \mathcal{M}\mathcal{P}^0 (y) c_\epsilon (e^0 (x)) + \mathcal{M}\mathcal{P}^0 (y, \omega) c_\epsilon (e^o (y)) \]

(2.3)

into the equation (2.2). The above summation is over \( p \) and \( g \); moreover, in \( u^1 (x, y) \) is a vector, the matrices \( \mathcal{N}P \) and \( \mathcal{M}\mathcal{P}^0 \) have vector components, i.e., the right-hand side is a linear combination of these vectors, with scalar coefficients \( c_\epsilon (u^0) \).

Integration by parts of the variational formulation showed that the strong form of the variational formulation for \( \mathcal{M}\mathcal{P}^0 \) seeks a solution such that

\[
\begin{align*}
div (K_M (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0)) &= 0 \quad \text{in } \mathcal{Y} \\
\mathcal{B}^1 (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0) &= \mathcal{A}^1 (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0) \nu \quad \text{on } \partial \mathcal{Y} \cap \partial \mathcal{Y}^1
\end{align*}
\]

\[ [\mathcal{M}\mathcal{P}^0] = 0, \quad \text{on } \partial \mathcal{Y} \cap \partial \mathcal{Y}^1 \]

where

\[
K_M = \mathcal{B}^1 (1 - \theta) \mathcal{A}^1
\]

(2.5)

Similarly, the strong form of the variation equation for \( \mathcal{M}\mathcal{P}^0 \) is to find a solution of

\[
\begin{align*}
div (K_M (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0)) &= -div (K_M c_\epsilon (\mathcal{M}\mathcal{P}^0)) \quad \text{in } \mathcal{Y} \\
\mathcal{B}^1 (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0) &= (\mathcal{A}^1 + \mathcal{B}^1) (\mathcal{M}\mathcal{P}^0 + \mathcal{E} \mathcal{P}^0 + \mathcal{C} \mathcal{P}^0) \nu \quad \text{on } \partial \mathcal{Y} \cap \partial \mathcal{Y}^1
\end{align*}
\]

\[ [\mathcal{M}\mathcal{P}^0] = 0, \quad \text{on } \partial \mathcal{Y} \cap \partial \mathcal{Y}^1 \]

where

\[
K_M = \mathcal{B}^1 (1 - \theta) (\mathcal{A}^1 + \mathcal{B}^1)
\]

(2.7)

The problems (2.5) and (2.7) are uniquely solvable for each \( \omega > 0 \). To show this, observe that \( \mathcal{A}^1, \mathcal{B}^1, \) and \( \mathcal{B}^1 \) (but not \( \mathcal{A}^1 \)) are strongly elliptic. Therefore \( K_M \) and \( K_M \) given by, respectively, (2.5), (2.7) are strongly elliptic for each \( \omega > 0 \). Now apply the complex version of the Lax-Milgram theorem to conclude.

3. ISOTROPIC CASE

In this section we write out explicit forms of the solutions assuming that the trabeculae are isotropic, i.e., we assume that in \( \mathcal{Y} \) (see Fig. 1)

\[
(\mathcal{A}^1 e_\epsilon)_{ij} = \delta_{ij} e_\epsilon + (\delta_{ij} e_\epsilon + 2\delta_{ij} e_\epsilon) e_\epsilon
\]

(3.1)

\[
(\mathcal{B}^1 e_\epsilon)_{ij} = \delta_{ij} e_\epsilon + 3\delta_{ij} e_\epsilon + 2\delta_{ij} e_\epsilon
\]

(3.2)

\[
(\mathcal{A}^1 e_\epsilon)_{ij} = \mathcal{A}^1 e_\epsilon + \mathcal{B}^1 e_\epsilon
\]

(3.3)
We may express these as matrix equations:

\[(\lambda + 2\mu)N^{\text{xx}} + 2\mu \Delta N^{\text{xx}} = 0 \text{ in } Y_s \quad (3.5)\]

\[(\eta + 2\mu)N^{\text{xx}} + 2\eta \Delta N^{\text{xx}} = 0 \text{ in } Y_f \quad (3.10)\]

where \(H\) is the Hessian operator, i.e.,

\[
H := \begin{pmatrix}
\frac{\partial^2}{\partial y_1^2} & \frac{\partial^2}{\partial y_1 \partial y_2} \\
\frac{\partial^2}{\partial y_2 \partial y_1} & \frac{\partial^2}{\partial y_2^2}
\end{pmatrix}
\]

We now turn to determining the matrix solutions \(N^{\text{xx}}\). To this end, we introduce \(Q^{\text{xx}} = N^{\text{xx}} + N^{\text{yy}}\), solve the problems for \(Q^{\text{xx}}\), and then obtain the solutions for \(N^{\text{xx}}\). First we compute the terms \((A' + iwB')Q^{\text{xx}}\) and \((A' + iwB')Q^{\text{yy}}\):

\[
(A' + iwB')Q^{\text{xx}} = \psi_0 \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_1 \partial y_2} \right) + 2iw\omega\psi_0
\]

\[
(A' + iwB')Q^{\text{yy}} = \psi_0 \left( \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_1 \partial y_2} \right) + 2iw\omega\psi_0
\]

This leads to the following equation holding in \(Y_f\):

\[
iu_G\Delta Q^{\text{xx}} + \left( \psi'_0 + iw\epsilon_0 + iu_G \right)HQ^{\text{xx}} = 0 \text{ in } Y_f \quad (3.12)
\]

On the other hand, in the solid part \(Y_s\), we have

\[
(A_s'Q^{\text{xx}})_{ij} = \psi_0 \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_1 \partial y_2} \right)
\]

\[
(A_s'Q^{\text{yy}})_{ij} = \psi_0 \left( \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_1 \partial y_2} \right)
\]

which yields

\[
(iu_G)^2\Delta Q^{\text{xx}} + (\eta + 2\mu + iw(\lambda + \mu))HQ^{\text{xx}} = 0 \text{ in } Y_s \quad (3.13)
\]

together with the following transmission conditions

\[
\gamma'_s \left[ (\lambda + 2\mu)Q^{\text{xx}} + 2\mu \Delta Q^{\text{xx}} \right]_{ij} + 2iu_G\omega\left( \frac{\partial}{\partial y_1}Q^{\text{xx}} + \frac{\partial}{\partial y_2}Q^{\text{yy}} \right) = 0
\]

\[
\gamma'_s \left[ (\eta + 2\mu)Q^{\text{xx}} + 2\eta \Delta Q^{\text{xx}} \right]_{ij} + 2iu_G\omega\left( \frac{\partial}{\partial y_1}Q^{\text{xx}} + \frac{\partial}{\partial y_2}Q^{\text{yy}} \right) = 0
\]

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4. THE EFFECTIVE EQUATIONS

To obtain the effective equations, we substitute (2.3) into (2.1) and collect terms containing the same components of $e(u^0)$:

$$\int \rho F \cdot \phi = - \omega^2 \int \rho u^0 \cdot \phi$$

$$+ \int \left( \int \left( (A^V + i \omega B^V) \cdot (F P + e(N^E + M^M)) \right) \cdot e(u^0) \cdot \phi \right) \, dx$$

$$+ \int \left( (1 - \theta) \cdot (A^L + i \omega B^L) \cdot (E^M + e(N^E + M^M)) \right) \cdot e(u^0) \cdot \phi \, dx$$

$$\times x \cdot (u^0)_{eq} \cdot e(\phi), \quad \forall \phi \in H^1_0(\Omega)^n$$

In (4.1), consider separately integrals over $\mathcal{Y}$ and recall that $N^E$ are independent of $\omega$. This allows us to separate the following three groups of terms: terms independent of $\omega$, terms that are linear in $\omega$, and the rest of the terms. This yields

$$\int \theta (A^k + i \omega B^k)_{ik} \cdot (E^M + e(N^E + M^M))_{ik}$$

$$+ \int (1 - \theta) (A^k + i \omega B^k)_{ik} \cdot (E^M + e(N^E + M^M))_{ik}$$

$$= \Lambda_{ik} + i \omega B_{ik} \cdot e(u^0) + G_{ik}(u^0)$$

(4.2)

where the effective materials tensors $A$, $B$, and $C$ are defined by

$$\Lambda_{ik} = \int \theta (A^k + i \omega B^k)_{ik} \cdot (E^M + e(N^E + M^M))_{ik}$$

(4.3)

$$\beta_{ik} = (1 - \theta) (A^k + i \omega B^k)_{ik} \cdot (E^M + e(N^E + M^M))_{ik}$$

(4.4)

$$G_{ik} = \int [(A^k + i \omega B^k)_{ik} + (1 - \theta) (A^k + i \omega B^k)_{ik}] \cdot (E^M + e(N^E + M^M))_{ik}$$

(4.5)

Combining (4.2) and (4.5) we obtain the weak formulation of the effective equation

$$\int \left( \int \left( \left(1 + \omega B^C \right) e(u^0) \cdot e(\phi) \right) - \omega^2 \rho u^0 \cdot \phi \right) \, dx$$

$$+ \int \left( \left(1 + \omega B^C \right) e(u^0) \cdot e(\phi) \right) \, dx$$

The proceeding can be summarized as the following.

Theorem 1. Let $u^0$ be the unique solution in $H^2_0(\Omega)$

$$u^0 = 0 \quad \text{on} \quad \partial \Omega$$

where $\mathcal{C}$ denotes the second order partial differential operator.

$$\mathcal{C} u^0 = - \nabla \cdot \left( \left(1 - \theta^2 \right) + \theta^2 \sigma \cdot e(u^0) \right) - \omega^2 \rho u^0$$

(4.6)

Then, there exists a subsequence $u^0$, not relabeled, such that $\{u^0\}$ converges weakly in $H^2_0(\Omega)$ to a limit $u^0 \in H^2_0(\Omega)$, and $\{u^0\}$ converges weakly in $L^2(\Omega)$ to $u^0$. The pair $\{u^0, \phi\}$ is a weak solution of the homogenized equation

$$\mathcal{C} u^0 = F_p$$

(4.7)

$$u^0 = 0 \quad \text{on} \quad \partial \Omega$$

where $\mathcal{C}$ denotes the homogenized operator such that

$$\mathcal{C} u = - \nabla \cdot \left( \left(1 - \theta^2 \right) \sigma \cdot e(u^0) + \theta^2 \sigma \cdot e(u^0) \right) - \omega^2 \rho u^0$$

The effective constant tensors $A$ and $B$ are defined in

(4.3), (4.4), respectively. The effective frequency dependent tensor $C(u^0)$ is defined in (4.5). The vectors $N^E$, $M^M$ that appear in (4.3)-(4.5) are solutions of the auxiliary cell problems (2.5), (2.7), respectively.

The homogenized behavior of the slightly compressible viscous fluid in the elastic porous medium is described by the equations (4.8), which are the equations of linear viscoelasticity, i.e., our mixture of a slightly compressible viscous fluid and an elastic solid behaves on average as a single-phase viscoelastic material. The effective tensors $A$, $B$, and $C$ characterize, respectively, elastic moduli, viscous moduli, and long-time relaxation moduli of the effective material.
9. NUMERICAL EXPERIMENTS

Using the physical values given in Table 1 and the value for λ, which is computed by the formula below:

\[ \lambda = K_b \frac{2}{1} \mu + \frac{(K_v - K_b)^2 - 2K_v(K_v - K_b) + 2K_v^2}{D - K_v} \]  

where

\[ D = K_v(1 + \beta(\mu/K_v - 1)) \]

we were able to compute the coefficients in the effective equations from the cell problem solutions \( N^{10} \) and \( M^{10} \) (p, q = 1, 2). The equations for \( N^{10} \) and \( M^{10} \) were discretized using a second-order finite difference scheme, and the resulting linear systems were solved numerically by a direct method. The numerical model was tested in simple cases. We show two typical coefficients \( N^{11} \) and \( N^{12} \) in Fig. 2 and Fig. 3 respectively. These numerical results were then used to compute the effective coefficients \( A_{ij}, E_{ij}, \) and \( C_{ij} \). Numerical quadrature, such as the trapezoidal rule, was used for these computations. Typically in our simulations, the unit cell was specified of length \( L = L \) in each direction, the trabecular frame was \( 2/3 \) of the total length of the unit cell in each direction, and the spatial resolution was \( \Delta L = 1.015^4 \). It is important to realize that these coefficients are in themselves only used to compute the effective constant coefficients appearing in the effective Eq. (4.9).

If we introduce the notation

\[ \mathcal{E}_{ijkl} = A_{ijkl} + \omega E_{ijkl} + C_{ijkl}(x) \]

then the effective equations, in the original \( x \)-coordinates, take the form

\[ \mathcal{E}_{ijkl} \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \omega^2 u_k = 0 \]

which in component form becomes
To interpret the meaning of our bone-coefficients, it is useful to compare those introduced by [6,7]. The Biot equations for the propagation of acoustic disturbances in a porous media were obtained using mixture theory. Identification of some our observations with Biot's physical parameters would give a hint as to which will play a significant role for the inverse problem. However, we must add the disclaimer that the Biot model was not meant for uniphasic vibrations, and our comparison becomes specious as it reduces to an elliptic system having only a principal part. Nevertheless, a comparison shows that our observations agree with the same order of magnitude. For simplicity, we consider the case where all parameters are constant in the Biot model. This leads to the system (5.3) below:

\[ \nabla^2 \mathbf{u} + \nabla(\|\mathbf{\mu} + \mathbf{v}\|) + \mathbf{Q}_1 = \frac{\mathbf{p}_1^2}{\mathbf{Q}_1} \left( \mathbf{p}_1 \mathbf{u} + \mathbf{p}_2 \mathbf{U} \right) + b \frac{\partial}{\partial \mathbf{u}(\mathbf{u} - \mathbf{U})} \]

(5.3)

\[ \nabla\mathbf{Q}_1 \mathbf{R} \mathbf{e} \mathbf{c} = \frac{\mathbf{p}_2}{\mathbf{Q}_1} \left( \mathbf{p}_1 \mathbf{u} + \mathbf{p}_2 \mathbf{U} \right) - \frac{\partial}{\partial \mathbf{u}(\mathbf{u} - \mathbf{U})} \]

where the coefficients \( \mathbf{\mu} \) and \( \lambda \) are Landé coefficients and where \( \omega = \nabla \mathbf{u}, \mathbf{c} = \nabla \mathbf{U} \).

The form of the dissipation parameter \( \lambda \) is complicated but not necessary for the uniphasic case. When the medium is undergoing time-harmonic oscillations of angular frequency \( \omega \), this becomes

\[ \nabla^2 \mathbf{u} + \nabla(\|\mathbf{\mu} + \mathbf{v}\|) + \mathbf{Q}_1 = \omega^2 (\mathbf{p}_1 \mathbf{u} + \mathbf{p}_2 \mathbf{U}) + i\nu \mathbf{u}(\mathbf{u} - \mathbf{U}) \]

(5.4)

\[ \nabla\mathbf{Q}_1 \mathbf{R} \mathbf{e} \mathbf{c} = -\omega^2 (\mathbf{p}_1 \nabla \mathbf{V} + \mathbf{p}_2 \nabla \mathbf{U}) - i\nu \mathbf{u}(\mathbf{u} - \mathbf{U}) \]

For the case of uniphasic oscillations, this reduces to a single equation

\[ \nabla^2 \mathbf{u} + \nabla(\|\mathbf{\mu} + \mathbf{v}\| + \mathbf{Q}_1 \mathbf{R} \mathbf{e} \mathbf{c}) \mathbf{u} = 0 \]

(5.5)

In component form, this leads to

\[ \mathbf{u} \Delta \mathbf{u} + \mathbf{u}^2 + \mathbf{Q}_1 \mathbf{R} \mathbf{e} \mathbf{c} \mathbf{u} = 0 \]

In component form, this leads to

\[ \mathbf{u} \Delta \mathbf{u} + \mathbf{u}^2 + \mathbf{Q}_1 \mathbf{R} \mathbf{e} \mathbf{c} \mathbf{u} = 0 \]
<table>
<thead>
<tr>
<th>Effective Elasticity Coefficients</th>
<th>Effective Viscosity Coefficients</th>
<th>Effective Relaxation Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{1111} = 4.58487e+09$</td>
<td>$B_{1111} = 1341.06$</td>
<td>$C_{1111} = 3.68323e+09 + 2.86725e+09i$</td>
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</tbody>
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<td>$B_{1112} = 356.247$</td>
<td>$C_{1112} = 7.2853e+08 + 7.24812e+08i$</td>
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<td>$B_{1112} = 0.0522359$</td>
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<td>$B_{1112} = 0.0522359$</td>
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</table>

<table>
<thead>
<tr>
<th>Effective Elasticity Coefficients</th>
<th>Effective Viscosity Coefficients</th>
<th>Effective Relaxation Coefficients</th>
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<tbody>
<tr>
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<td>$C_{1122} = 3.0628e+08 + 2.78353e+09i$</td>
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<td>$C_{1122} = 332428 + 1.16434e+09i$</td>
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<td>$B_{1122} = 0.0393856$</td>
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<td>$B_{1122} = 356.175$</td>
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<td>$B_{1222} = 0.00621678$</td>
<td>$C_{1222} = -191283 + 993380i$</td>
</tr>
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<td>$A_{1222} = -7.16565e-07$</td>
<td>$B_{1222} = 0.00621678$</td>
<td>$C_{1222} = -191283 + 993380i$</td>
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<td>$B_{1222} = 1333.17$</td>
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</table>
\[
\mu \Delta u_{1} + \left( \frac{\lambda + \mu}{2} + \frac{Q}{12} \right) \frac{\partial^{2} u_{1}}{\partial x_{1}^{2}} + \frac{\partial^{2} u_{2}}{\partial x_{2}^{2}} = 0
\]

or

\[
\left( 2\mu + Q \frac{\partial^{2}}{\partial x_{1}^{2}} + \mu \frac{\partial^{2}}{\partial x_{2}^{2}} \right) u_{1} + \frac{\partial u_{1}}{\partial x_{1}} = 0
\]

The parameter \( \mu \), the complex frame shear modulus, is measured. The other parameters \( \lambda, R, \) and \( Q \) occurring in the constitutive equations are calculated from the measured or estimated values of the parameters given in Table 1 using the formulas

\[
R = \frac{K_{2}^{2}}{D - K_{5}}
Q = \frac{K_{5} (1 - \beta) (K_{2} - K_{5})}{D - K_{5}}
\]

In the Biot theory, the bulk and shear moduli \( K_{4} \) and \( K_{5} \) are often given imaginary parts to account for frame inelasticity. Here \( \rho_{12} \) and \( \rho_{22} \) are density parameters for the solid and fluid, and \( \rho_{13} \) is a density coupling parameter. These are calculated from the inputs of Table 5 using the formulas

\[
\rho_{13} = (1 - \beta) \rho_{v} - \beta (\rho_{f} - m \beta)
\]

\[
\rho_{12} = \beta (\rho_{f} - m \beta)
\]

\[
\rho_{22} = m \beta^{2}
\]

where

\[
m = \frac{\alpha}{\beta}
\]

This suggests that we might try to associate, at least up to orders of magnitude, out coefficients \( E_{ij} \) with the Biot coefficients. The general form of the coefficients for the Biot uniphasic model are

\[
E_{111} = 2\mu + Q \frac{\partial^{2}}{\partial x_{1}^{2}} + \mu \frac{\partial^{2}}{\partial x_{2}^{2}} (Q + R)
\]

\[
E_{122} = \mu + \lambda + Q \frac{\partial^{2}}{\partial x_{1}^{2}} + \mu \frac{\partial^{2}}{\partial x_{2}^{2}} (Q + R)
\]

\[
E_{112} = \mu + \lambda + Q \frac{\partial^{2}}{\partial x_{1}^{2}} + \mu \frac{\partial^{2}}{\partial x_{2}^{2}} (Q + R)
\]

\[
E_{311} = \mu + \lambda + Q \frac{\partial^{2}}{\partial x_{1}^{2}} + \mu \frac{\partial^{2}}{\partial x_{2}^{2}} (Q + R)
\]

\[
E_{322} = \mu + \lambda + Q \frac{\partial^{2}}{\partial x_{1}^{2}} + \mu \frac{\partial^{2}}{\partial x_{2}^{2}} (Q + R)
\]

6. CONCLUSION

We have shown that the method of two scale convergence may be used to compute effective bone parameters in the monophasic case that are comparable to those found in the Biot model. We are planning to use this model for the inverse problem for determining the bone coefficients. Further work is planned to complete the biphysic case. The biphysic is more suitable for comparison with the Biot model. Homogenization leads to systems similar to the Biot model but having more coefficients. Changing some of the physical assumptions about the fluid solid interaction may lead to new models more suitable for the ultrasound interrogation of bone.

ACKNOWLEDGMENTS

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TABLE 5. Additional parameters needed in the Biot model when $\beta = 0.76$

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_f = 5.304000 \times 10^4$</td>
<td>density</td>
</tr>
<tr>
<td>$\rho_v = 60.00000$</td>
<td>density</td>
</tr>
<tr>
<td>$\eta_v = 8.2000000 \times 10^4$</td>
<td>density</td>
</tr>
<tr>
<td>$\lambda = 2.02034069 \times 10^9$</td>
<td>Lamé coefficient</td>
</tr>
<tr>
<td>$E = 1.459870103 \times 10^{13}$</td>
<td>solid-fluid coupling coefficient</td>
</tr>
<tr>
<td>$Q = 1.5680000 \times 10^{11}$</td>
<td>solid-fluid coupling coefficient</td>
</tr>
</tbody>
</table>

REFERENCES


