Chapter 6
An Asymptotic Method to a Financial Optimization Problem

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Abstract This paper studies the borrower's optimal strategy to close the mortgage when the volatility of the market investment return is small. Integral equation representation of the mortgage contract value is derived, then used to find the numerical solution of the free boundary. The asymptotic expansions of the free boundary are derived for both small time and large time. Based on these asymptotic expansions two simple analytical approximation formulas are proposed. Numerical experiments show that the approximation formulas are accurate enough from practitioner's point of view.

Keywords Mortgage prepayment · Asymptotic analysis · Numerical solution · Analytical approximation

6.1 Introduction

Consider a mortgage with a fixed interest rate of c (year^{-1}). Assume that the underlying risk free rate following the CIR model [1], which says $dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$, where k, θ, σ are positive constants. According to standard mathematical finance theory (see [4, 8–10], for instance), the value of the mortgage contract $V(x,t)$ at any specified t, the time left to the expiry of the contract, and the corresponding interest rate x , when it is not optimal for prepayment, satisfies

$$\frac{\partial V}{\partial t} - \frac{\sigma^2}{2} x \frac{\partial^2 V}{\partial x^2} - k(\theta - x)\frac{\partial V}{\partial x} + xV = m; \tag{6.1}$$

and when the borrower decides to terminate the contract prematurely at time t, he needs to pay the mortgage loan balance

$$M(t) = \frac{m}{c}\left[1 - e^{-ct}\right], \tag{6.2}$$

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where m denotes the continuous mortgage payment rate, i.e., the borrower pays mdt (dollars) to the mortgage contract holder (the lender) for each time period dt. Mathematically we have a free boundary problem where the free boundary $x = h(t)$ defines the optimal market interest rate level at which the borrower should terminate the contract. For the continuation region where $x > h(t)$, the contract is in effect and the value of the contract satifies (6.1). For the early exercise region where $x \leq h(t)$, the contract is closed and the lender gets back the loan balance of $M(t)$. Because it is the borrower, rather than the lender, who is a proactive player of the game and has the choice to act in response to the market, so the value of the contract is always less or equal than the loan balance. Thus the free boundary is where the value of the contract $V(x,t)$ first reaches the value of the mortgage loan balance $M(t)$. It is easy to show, using the free of arbitrage argument, that the free boundary starts from c, i.e., $h(0) = c$. And because of the smooth patch is needed for the regularity of the problem, we have the derivative of $V(x,t)$ must be 0 on $h(t)$. Lastly, it is trivially true that $V(x,0) = 0$, which says that the value of the contract, when the contract is expired, must be 0. Putting all these condition together, we formulate the problem as follows: for $\forall x \geq 0$ and $t > 0$, find $V(x,t)$ and $h(t)$ such that

$$\begin{cases} \mathbf{L}(V) = m, & \text{for} \quad x > h(t), t > 0 \\ V = \dfrac{m}{c}[1 - e^{-ct}], & \text{for} \quad x \leq h(t), t > 0 \\ \dfrac{\partial V}{\partial x}(h(t),t) \equiv 0 & \\ V(x,0) = 0, & \text{for all} \quad x \geq 0 \\ h(0) = c & \end{cases} \tag{6.3}$$

where the differential operator $\mathbf{L}$ is defined as

$$\mathbf{L}(V) = \frac{\partial V}{\partial t} - \frac{\sigma^2}{2} x \frac{\partial^2 V}{\partial x^2} - k(\theta - x)\frac{\partial V}{\partial x} + xV \tag{6.4}$$

Because of the important role played by mortgage backed securities in real economy, there has been continuing interest in mortgage pricing and related problems, especially the prepayment strategies for mortgage borrowers. Most of the studies, such as [2, 5], are from option-theoretical viewpoint. A similar problem with underlying interest rate following Vasicek model was recently studied with variational integral equation approach in [3, 7]. In this paper, we focus on the situation where the volatility σ is small. Such an assumption is reasonable for the long term real economy. More discussions on parameter estimation for risk free market return can be found in [6].

Here we first derive the integral representation of the solution with the free boundary embedded, then prove the monotonocity and boundedness of the free boundary, then design an effective iteration scheme to solve the problem numerically. Based on the asymptotic analysis, we drive two analytical approximation formulas for the optimal prepayment boundary. Numerical simulations are carried out to validate our approximation formulas.

6.2 Integral Representation of the Solution

We first derive the characteristic solution. When $\sigma \to 0$, the PDE (6.1) reduces to

$$\frac{\partial V}{\partial t} - k(\theta - x)\frac{\partial V}{\partial x} + xV = m. \tag{6.5}$$

Lemma 6.2.1 *The characteristic solution associated with (6.5) is*

$$V(X(t), t) = me^{-\theta t - \frac{X(t)-\theta}{k}} \int_0^t e^{\theta\tau + \frac{X(\tau)-\theta}{k}} d\tau, \tag{6.6}$$

where

$$X(t) = \theta + (X_0 - \theta)e^{kt} \tag{6.7}$$

for each given $X(0) = X_0$.

Proof. Starting with each intial point $(X(0), 0)$, $X(0) = X_0 \geq 0$, we define the characteristic curve related to (6.5) as

$$\frac{\partial X}{\partial t} = -k(\theta - X), \quad X(0) = X_0,$$

which gives

$$X(t) = \theta + (X_0 - \theta)e^{kt}. \tag{6.8}$$

Now on each characteristic curve, we have the following ODE for $V(X(t), t)$ defined as

$$\frac{dV}{dt} + (\theta + (X_0 - \theta)e^{kt})V = m$$

or equivalently

$$\frac{d}{dt}\left\{Ve^{\theta t + \frac{X_0-\theta}{k}e^{kt}}\right\} = me^{\theta t + \frac{X_0-\theta}{k}e^{kt}}$$

the solution of which is

$$V = me^{-\theta t - \frac{X_0-\theta}{k}e^{kt}} \int_{t^*}^t e^{\theta\tau + \frac{X_0-\theta}{k}e^{k\tau}} d\tau \tag{6.9}$$

Because of the requirement of $V(X(0), 0) = 0$, we have $t^* = 0$.

Lemma 6.2.2 *The solution to (6.5) is given by*

$$V(x, t) = me^{-\frac{x-\theta}{k}} \int_0^t e^{-\theta s + \frac{x-\theta}{k}e^{-ks}} ds, \tag{6.10}$$

it is strictly decreasing in x*, ranges from* $\lim_{x\to-\infty} V(x,t) = \infty$ *to* $\lim_{x\to\infty} V(x,t) = 0$.

Proof. Reformulate the equation of $V(X(t),t)$ in (6.6), we have

$$V(X(t),t) = me^{-\theta t - \frac{X(t)-\theta}{k}} \int_0^t e^{\theta\tau + \frac{X(\tau)-\theta}{k}} d\tau$$
$$= me^{-\frac{X(t)-\theta}{k}} \int_0^t e^{-\theta s + \frac{X(t)-\theta}{k} e^{-ks}} ds.$$

Write $x = X(t)$, we have the desired form of the solution as in (6.10). To validate the monotonocity of V in x, we note that, in the set where V satisfies (6.5), $V_x := \frac{\partial V}{\partial x}$ satisfies the differential inequality

$$\frac{\partial V_x}{\partial t} - k(\theta - x)\frac{\partial V_x}{\partial x} + (x+k)V_x = -V < 0.$$

The limits of V as x approaches $\pm\infty$ are the results of a simple computation.

6.3 Properties of the Free Boundary

In this section we shall show the monotonocity of the free boundary $h(t)$ and the existence of $\lim_{t\to\infty} h(t)$ and $\lim_{t\to 0_+} h(t)$, namely, we shall prove the following theorem:

Theorem 6.1. *If* $c < \theta$*, then* $h(t)$*, starts from* $h(0) = c$*, is continuous and monotonously decreasing in* $[0,\infty)$*, and is lower bounded. If* $c > \theta$*, then* $h(t)$*, starts from* $h(0) = c$*, is continuous and monotonously increasing in* $[0,\infty)$*, and is upper bounded.*

Proof. The theorem is a summary of the following Lemmas 6.3.1–6.3.6 and Corollaries 1–2. The proof is organized as follows: we first show the existence, uniqueness, and continuity of $h(t)$, except possibly for $t = 0$, then show the boundedness of $h(t)$ both from below and above, then the monotonocity, and lastly we find the limit of $h(t)$ at $t = 0$.

Lemma 6.3.1 *For each* $t \geq 0$*, h(t) exists and is unique. h(t) is continuous for all* $t \geq 0$ *except possibly at* $x = 0$.

Proof. The existence and uniqueness is naturally concluded from Lemma 6.2.2. The continuity of $h(t)$ for $t > 0$ is a consequence of the continuity of V in x. The only thing left to validate is $\lim_{t\to 0_+} h(t) = c$, which is to be done after we prove the boundedness of $h(t)$.

Lemma 6.3.2 *If* $c > \theta$*,* $\sigma \to 0$*, the free boundary* $h(t)$ *in (6.3) is lower bounded by* c*, i.e.,*

$$h(t) > c \quad \forall t > 0. \tag{6.11}$$

Proof. Because $V(X(t), t)$ is monotoneously decreasing (to 0) in $X(t)$ for fixed $t > 0$, i.e., $\frac{\partial V}{\partial X} < 0, \forall t > 0$, which is shown in the Lemma 6.2.2, it suffices to show $V(c,t) > M(t)$, where $M(t) = \frac{m}{c}\left[1 - e^{-ct}\right]$ is the contract value on the free boundary. Recall $V(x,t) = e^{-\frac{x-\theta}{k}} \int_0^t e^{-\theta s + \frac{x-\theta}{k} e^{-ks}} ds$ (hereafter, we assume, WLOG, $m = 1$), we have

$$V(c,t) = e^{-\alpha} \int_0^t e^{(k\alpha - c)s + \alpha e^{-ks}} ds,$$

by letting $\frac{c-\theta}{k} = \alpha$. Now, noticing $\alpha > 0$, we have

$$V(c,t) - M(t) = e^{-\alpha} \left\{ \int_0^t e^{-cs} \left[e^{k\alpha s + \alpha e^{-ks}} - e^{\alpha} \right] ds \right\}.$$

Because

$$\begin{aligned} k\alpha s + \alpha e^{-ks} &= \alpha(ks + e^{-ks}) \\ &> \alpha \end{aligned}$$

We have

$$V(c,t) - M(t) > 0$$

and thus completes the proof.

Corollary 6.1. *If $c < \theta$, $\sigma \to 0$, the free boundary $h(t)$ in (6.3) is upper bounded by c, i.e.,*

$$h(t) < c \quad \forall t > 0.$$

Proof. Follow the same procedure of the above proof except this time $\alpha < 0$, and thus changes the sign of $V(c,t) - M(t)$.

Lemma 6.3.3 *If $c > \theta$, then $h(t)$ is monotonously increasing in t, i.e., $h'(t) > 0, \forall t > 0$, and* $\lim_{t\to\infty} h'(t) = 0$.

Proof. Knowing that $V(h(t),t) = \frac{1}{c}\left[1 - e^{-ct}\right]$, we have, for $\forall t > 0$,

$$e^{-\frac{h(t)-\theta}{k}} \int_0^t e^{-\theta s + \frac{h(t)-\theta}{k} e^{-ks}} ds = \frac{1}{c}\left[1 - e^{-ct}\right],$$

or equivalently,

$$\int_0^t e^{-\theta s - \frac{h(t)-\theta}{k}[1-e^{-ks}]} ds = \frac{1}{c}\Big[1 - e^{-ct}\Big].$$

Differentiating it with respect to t, we get

$$-\frac{h'(t)}{k}\int_0^t e^{-\theta s - \frac{h(t)-\theta}{k}[1-e^{-ks}]}\Big[1-e^{-ks}\Big] ds$$

$$+ e^{-\theta t - \frac{h(t)-\theta}{k}[1-e^{-kt}]} = e^{-ct}. \tag{6.12}$$

Notice that the definite integral in above equation is strictly positive. If the second term is strictly greater than e^{-ct}, then $h'(t) > 0$ is necessary for the above equation to hold. Now the previous Lemma 6.3.2 tells us that $h(t) > c$, hence

$$h(t) - \theta > 0,$$

and also

$$0 < 1 - e^{-kt} < kt,$$

we have

$$\frac{h(t)-\theta}{k}[1-e^{-kt}] < \frac{c-\theta}{k}kt = ct - \theta t.$$

So

$$e^{-\theta t - \frac{h(t)-\theta}{k}[1-e^{-kt}]} > e^{-\theta t-(ct-\theta t)} = e^{-ct},$$

which is the desired inequality leading the monotonocity of the $h(t)$. Lastly, if we let $t \to \infty$ in (6.3), we have both the righthand side and the second term in the left side vanish, thus forces the first term in the left side vanish too. But the definite integral itself is strictly positive, so $\lim_{t\to\infty} h'(t) = 0$ becomes necessary, thus completes the proof.

Corollary 6.2. *If $c < \theta$, then $h(t)$ is monotonously decreasing in t, i.e., $h'(t) < 0, \forall t > 0$, and $\lim_{t\to\infty} h'(t) = 0$.*

Lemma 6.3.4 *If $c > \theta$, then $lim_{t\to\infty} h(t)$ exists. For $\forall$ fixed $\epsilon > 0$, $lim_{t\to\infty} h(t) < \big[c - \theta + \frac{c}{\theta}e^{-(\epsilon\theta+1)}\big]\frac{\epsilon k}{1-e^{-\epsilon k}} + \theta$. In particular, $lim_{t\to\infty} h(t) < c + \frac{c}{\theta}$*

Proof. Let $\lim_{t\to\infty} h(t) = h^*$. Knowing the contract value at t infinity is $\frac{1}{c}$, we wish to balance the following parametric integral of h^*

$$\frac{1}{c} = \int_0^\infty e^{-\theta s - \frac{h^*-\theta}{k}[1-e^{-ks}]} ds.$$

The boundedness of h^* is immediate simply because $\lim V(x,t)_{x\to\infty} \to 0$. Here we are interested in finding a particular value of the bound. Fix $\epsilon > 0$, let $1-e^{-k\epsilon} = \lambda$. Notice that $1 - e^{-ks} > \frac{\lambda}{\epsilon}$ for $0 < s < \epsilon$ and $1 - e^{-ks} > \lambda$ for $s > \epsilon$, we have

$$\frac{1}{c} < \int_0^\epsilon e^{-\theta s - \frac{h^*-\theta}{k}\frac{\lambda}{\epsilon}s} ds + \int_\epsilon^\infty e^{-\theta s - \frac{h^*-\theta}{k}\lambda} ds = \frac{\theta + ye^{-(\theta+y)\epsilon}}{(\theta+y)\theta},$$

where $y := \frac{h^*-\theta}{k}\frac{\lambda}{\epsilon}$. Now we have

$$\frac{\theta(c-\theta)}{c} > \frac{\theta}{c}y - ye^{-(\theta+y)\epsilon},$$

since $c > \theta$. The condition $h^* > c > \theta$ here plays its role because otherwise $\theta + y$ is not necessarily positive. Notice that the function defined by $f(y) = ye^{-(\theta+y)\epsilon}$ achieves the absolute maximum of $e^{-(\epsilon\theta+1)}$at $y = \frac{1}{\epsilon}$, we have

$$\frac{\theta(c-\theta)}{c} > \frac{\theta}{c}y - e^{-(\epsilon\theta+1)}.$$

Correspondingly, we have

$$h^* < \left[c - \theta + \frac{c}{\theta}e^{-(\epsilon\theta+1)}\right]\frac{\epsilon k}{1-e^{-\epsilon k}} + \theta.$$

The righthand side of above inequality is continuous in ϵ. Take limit for $\epsilon \to 0$, we find $[c - \theta + \frac{c}{\theta}e^{-(\epsilon\theta+1)}]\frac{\epsilon k}{1-e^{-\epsilon k}} + \theta < c + \frac{c}{\theta}$.

Lemma 6.3.5 *If $c < \theta$, then $lim_{t\to\infty}h(t)$ exists. For $\forall$ fixed $\epsilon > 0$, $lim_{t\to\infty}h(t) > \theta\left(1 - \frac{k\epsilon}{1-e^{-k\epsilon}}\right) - \frac{k\theta}{c}\frac{1}{1-e^{-k\epsilon}}$.*

Proof. Again, we wish to balance the following parametric integral of h^*

$$\frac{1}{c} = \int_0^\infty e^{-\theta s - \frac{h^*-\theta}{k}[1-e^{-ks}]} ds.$$

Due to Lemma 6.2.2, the lower boundedness of h^* is apparent. Here we are interested in finding a particular value of the lower bound. Fix $\epsilon > 0$, let $1 - e^{-k\epsilon} = \lambda$. Notice that $1 - e^{-ks} > \frac{\lambda}{\epsilon}$ for $0 < s < \epsilon$ and $1 - e^{-ks} > \lambda$ for $s > \epsilon$, we have

$$\begin{aligned}\frac{1}{c} &> \int_0^\epsilon e^{-\theta s - \frac{h^*-\theta}{k}\frac{\lambda}{\epsilon}s} ds + \int_\epsilon^\infty e^{-\theta s - \frac{h^*-\theta}{k}\lambda} ds \\ &= \frac{\theta - ye^{-(\theta-y)\epsilon}}{(\theta-y)\theta},\end{aligned}$$

where $y := \frac{\theta-h^*}{k}\frac{\lambda}{\epsilon}$. If $\theta - y \geq 0$, then

$$\theta \geq (\theta - h^*) \frac{1 - e^{-k\epsilon}}{k\epsilon},$$

which is equivalent to

$$h^* \geq \theta \left(1 - \frac{k\epsilon}{1 - e^{-k\epsilon}}\right).$$

If $\theta - y < 0$ then we have

$$\begin{aligned}
\frac{1}{c} &> \frac{\theta - ye^{-(\theta-y)\epsilon}}{(\theta - y)\theta} \\
&= \frac{e^{(y-\theta)\epsilon} - 1}{\theta - y} + \frac{e^{(y-\theta)\epsilon}}{\theta} \\
&> \frac{\epsilon}{\theta}(y - \theta),
\end{aligned}$$

which gives

$$h^* \geq \theta \left(1 - \frac{k\epsilon}{1 - e^{-k\epsilon}}\right) - \frac{k\epsilon}{1 - e^{-k\epsilon}} \frac{\theta}{\epsilon c}.$$

Because we are looking for a lower bound, so we take the minimum of the two cases (for fixed $\epsilon > 0$), and conclude that

$$h^* \geq \theta \left(1 - \frac{k\epsilon}{1 - e^{-k\epsilon}}\right) - \frac{k\theta}{c} \frac{1}{1 - e^{-k\epsilon}}.$$

Lemma 6.3.6 *$h(t)$ is continuous for $t \in [0, \infty)$, in particular,* $\lim_{t\to 0_+} = c$.

Proof. Because of Lemma 6.3.1, the only thing left to be justified is $\lim_{t\to 0_+} = c$. For t small, $e^{-ks} = 1 - ks$, we have

$$\begin{aligned}
\lim_{t\to 0_+} V(h(t), t) &= \lim_{t\to 0_+} e^{-\frac{h(t)-\theta}{k}} \int_0^t e^{-\theta s + \frac{h(t)-\theta}{k} e^{-ks}} ds \\
&= \lim_{t\to 0_+} e^{-\frac{h(t)-\theta}{k}} \int_0^t e^{-\theta s + \frac{h(t)-\theta}{k}(1-ks)}
\end{aligned}$$

Because of the continuity and boundedness of $h(t)$, we can take limit of $\lim_{t\to 0_+} h(t)$ inside of the integral and arrive at

$$\lim_{t\to 0_+} V(h(t), t) = \frac{1}{\lim_{t\to 0_+} h(t)}$$

Compare this with the boundary value of $\frac{1}{c}[1 - e^{ct}]$, we have that $\lim_{t\to 0_+} h(t) = c$.

6.4 Numerical Solution of the Free Boundary

Since $\frac{\partial V}{\partial x} \neq 0$ we can use Newton method to solve for the free boundary iteratively. Define

$$Q(h) = e^{-\frac{h-\theta}{k}} \int_0^t e^{-\theta s + \frac{h-\theta}{k} e^{-ks}} ds - \frac{1}{c}\Big[1 - e^{ct}\Big],$$

and

$$f(h) = e^{-\frac{h-\theta}{k}} \left[-\frac{1}{k}\right] \int_0^t e^{-\theta s + \frac{h-\theta}{k} e^{-ks}} ds + e^{-\frac{h-\theta}{k}} \int_0^t e^{-\theta s + \frac{h-\theta}{k} e^{-ks}} \left[\frac{1}{k} e^{-ks}\right] ds,$$

our problem is to find h such that

$$Q[h](t) \equiv 0, \qquad \forall t \geq 0.$$

For fixed $t = T$, discretize $[0, T]$ uniformly into n subintervals by $t_0, t_1, t_2, ..., t_n$, where $t_0 = 0, t_n = T$. Start with $h(t_0) = c$ and assume $h(t_1), h(t_2), ..., h(t_{n-1})$ are known, to compute $h(t_n)$ with Newton's algorithm, we first assign a reasonable initial guess for $h(t_n)$ as

$$h^0(t_n) = h(t_{n-1}), \qquad n = 1;$$
$$h^0(t_n) = 2h(t_{n-1}) - h(t_{n-2}), \qquad n > 1.$$

For a given error tolerance level, say $Tole = 10^{-7}$, we have the following Newton's iteration scheme

$$h(t_n)^{new} = h(t_n)^{old} - \frac{Q(h(t_n)^{old})}{f(h(t_n)^{old})}.$$

After each step of iteration, a current error is recorded as

$$error(k) = h(t_n)^{new} - h(t_n)^{old}.$$

The iteration is kept running until an integer k is reached such that $error(k) < Tole$. To increase the accuracy of the numerical solution, one can increase N, the number of grids for partitioning the time interval $[0, T]$. For typical parameters with $T \leq 25$, our numerical simulations show that $N = 4096$ is large enough for achieving a solution with relative error less than 10^{-7}, where relative error is defined as the difference of numerical values of $h(T)$'s achieved with different N's. Figure 6.1 is a numerical plot of the free boundaries as we fix one set of parameters at a time

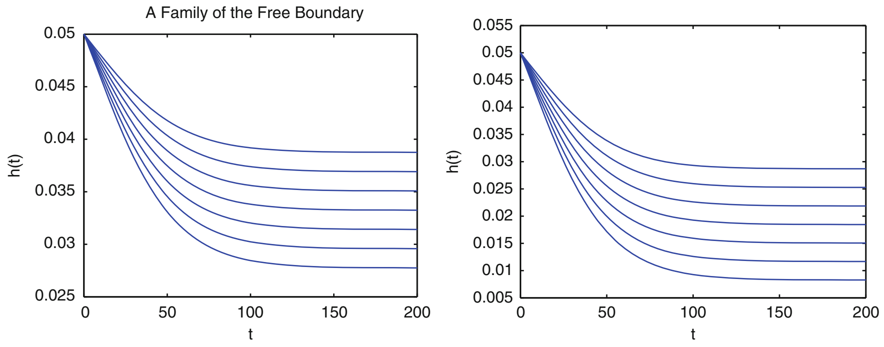

Fig. 6.1 c = 0.05, $k = 0.06, 0.07, ..., 0.12$ (*top to bottom*) $\theta = 0.06$ (*right*), 0.07 (*left*). The units for t and $h(t)$ are years and year^{-1}, respectively

6.5 Asymptotic Analysis of the Free Boundary

In this section, we derive asymptotic expansions of $h(t)$ for both small t and large t.

Theorem 6.2. *As* $t \to 0$, $h(t) \sim c + \alpha t$, *where* $\alpha = \frac{(c-\theta)k}{3}$.

Proof. We postulate that as $t \to 0$, $h(t) \sim c + \alpha t$, plug this into the contract value on $h(t)$, we have that, for t small,

$$V(h(t),t) = \int_0^t e^{-\theta s - \frac{c+\alpha t-\theta}{k}[1-e^{-ks}]} ds$$
$$= \int_0^t e^{\frac{(c-\theta+\alpha t)k}{2}s^2 - (c+\alpha t)s} ds$$

For $a, b > 0$, s small, we have the following Taylor expansion

$$e^{as^2-bs} = \left\{1 + as^2 + \frac{a^2s^4}{2!} + ...\right\}\left\{1 - bs + \frac{b^2s^2}{2!} - ...\right\}$$
$$= 1 - bs + \left(a + \frac{b^2}{2}\right)s^2 - \left(ab + \frac{b^3}{3!}\right)s^3 + o(s^3)$$

Integrating it term by term for small t, we have

$$\int_0^t e^{as^2-bs} ds = t - \frac{b}{2}t^2 + \frac{1}{3}\left(a + \frac{b^2}{2}\right)t^3$$
$$-\frac{1}{4}\left(ab + \frac{b^3}{3!}\right)t^4 + o(t^4).$$

In terms of our problem at hand, we have

$$V(h(t),t) = t - \frac{c+\alpha t}{2}t^2 + \frac{1}{3}\Big[\frac{c-\theta+\alpha t}{2} + \frac{(c+\alpha t)^2}{2}\Big]t^3 - o(t^3).$$

We want to match this, term by term, with the known expression of $V(h(t),t)$, which is

$$\frac{1}{c}[1-e^{-ct}] = t - \frac{c}{2}t^2 + \frac{1}{3!}c^2t^3 - o(t^3).$$

In the above two series, the coefficients for t and t^2 match each other automatically. To match the t^3 terms, we need to have

$$-\frac{\alpha}{2} + \frac{1}{3}\left(\frac{(c-\theta)k + c^2}{2}\right) = \frac{1}{6}c^2,$$

which gives

$$\alpha = \frac{(c-\theta)k}{3}.$$

Theorem 6.3. *There exist constants $h^* = \lim_{t\to\infty} h(t)$, $\rho_1 > 0$, and $\rho_2 > 0$ such that, as $t \to \infty$,*

$$h(t) \sim h^* - \rho_1 e^{-\theta t}, \quad \textit{if} \quad c < \theta,$$
$$h(t) \sim h^* + \rho_2 e^{-ct}, \quad \textit{if} \quad c > \theta,$$

where h^* is implicitly given by $M\left(1, \frac{\theta}{k}+1, -\frac{h^*-\theta}{k}\right) = \frac{\theta}{c}$, where $M(p,q,z)$ is the confluent hypergeometric function of the first kind of order p, q, and

$$\rho_1 = \frac{kc(h^*-\theta)e^{-\frac{h^*-\theta}{k}}}{\theta(h^*-c)},$$
$$\rho_2 = \frac{k(h^*-\theta)}{h^*-c}.$$

The existence and boundedness of h^* have been previously shown. The main idea to find the exact value of h^* is to use repeated integration by parts to express the contract value V at t infinity as a infinite series involving h^*, which turns out to be a confluent hypergeometric function. As in general, given $a, b, c > 0$, we have

$$\int_0^\infty e^{-ay+be^{-cy}}dy = -\frac{1}{a}\int_0^\infty e^{be^{-cy}}[e^{-ay}]'dy$$
$$= \frac{1}{a}e^b + (-1)\frac{bc}{a}\int_0^\infty e^{-(a+c)y+be^{-cy}}dy$$

Repeating the integration by parts, using the recursive identity

$$\int_0^\infty e^{-(a+nc)y+be^{-cy}}dy = -\frac{1}{a+nc}e^b + (-1)\frac{bc}{a+nc}\int_0^\infty e^{-(a+(n+1)y)+be^{-cy}}dy,$$

where the tail definite integral vanishes as $n \to \infty$, we have

$$\int_0^\infty e^{-ay+be^{-cy}}dy = \frac{1}{a}\sum_{n=1}^{n=\infty}(-1)^n \frac{b^n}{(a/c+1)(a/c+2)...(a/c+n)}e^b$$
$$= \frac{1}{a}M(1, a/c+1, -b).$$

In terms of our problems, this means

$$e^{-\frac{h^*-\theta}{k}e^{-ks}}\int_0^\infty e^{-\theta s+\frac{h^*-\theta}{k}e^{-ks}}ds = \frac{1}{\theta}M\left(1, \theta/k+1, -\frac{h^*-\theta}{k}\right).$$

At t infinity, we want

$$V(h(\infty), \infty) = e^{-\frac{h^*-\theta}{k}e^{-ks}}\int_0^\infty e^{-\theta s+\frac{h^*-\theta}{k}e^{-ks}}ds = \frac{1}{c},$$

which means

$$M\left(1, \frac{\theta}{k}+1, -\frac{h^*-\theta}{k}\right) = \frac{\theta}{c}.$$

To fully understand the asymptotic behavior of the free boundary as $t \to \infty$, we evaluate the limit of $h'(t)$ as $t \to \infty$. Start with the equation

$$\int_0^\infty e^{-\theta s-\frac{h(t)-\theta}{k}[1-e^{-ks}]}ds = \frac{1}{c}\Big[1-e^{-ct}\Big],$$

take derivative with respect to t along $h(t)$,

$$-\frac{h'(t)}{k}\int_0^t e^{-\theta s-\frac{h(t)-\theta}{k}[1-e^{-ks}]}\Big[1-e^{-ks}\Big]ds =$$
$$-e^{-\theta t-\frac{h(t)-\theta}{k}[1-e^{-kt}]} + e^{-ct},$$

we get

$$\frac{h'(t)}{k} = \frac{e^{-\theta t-\frac{h(t)-\theta}{k}[1-e^{-kt}]} - e^{-ct}}{\frac{1}{c}[1-e^{-ct}] - I},$$

where

$$I := \int_0^t e^{-\theta s - \frac{h(t)-\theta}{k}[1-e^{-ks}]} e^{-ks} ds,$$

which can be evaluated using integration by parts. Thus we get

$$\frac{h'(t)}{k} = \frac{(h(t)-\theta)\left\{e^{-\theta t - \frac{h(t)-\theta}{k}[1-e^{-kt}]} - e^{-ct}\right\}}{\frac{h(t)}{c}[1-e^{-ct}] + e^{-\theta t - \frac{h(t)-\theta}{k}[1-e^{-kt}]} - 1}.$$

When $c > \theta$, we can write the above equation into

$$h'(t) = F(t)e^{-\theta t},$$

where

$$F(t) = \frac{e^{-\frac{h(t)-\theta}{k}[1-e^{-kt}]} - e^{-(c-\theta)t}}{\frac{h(t)}{c(h(t)-\theta)}[1-e^{-ct}] + \frac{1}{h(t)-\theta}\left\{e^{-\theta t - \frac{h(t)-\theta}{k}[1-e^{-kt}]}\right\}}.$$

Since it has been shown that $\lim_{t\to\infty} h(t) = h^* > \theta$, it is straightforward to show that $F(t)$ is uniformly bounded in t and

$$\lim_{t\to\infty} F(t) = \frac{kc(h^*-\theta)e^{-\frac{h^*-\theta}{k}}}{h^*-c}.$$

Now we postulate

$$h(t) \sim h^* - \rho_1 e^{-\theta t}, \quad \text{if } c > \theta.$$

Compare the limit of $h'(t)$, we get

$$\rho_1 = \frac{kc(h^*-\theta)e^{-\frac{h^*-\theta}{k}}}{\theta(h^*-c)}.$$

On the other hand, if $c < \theta$, we can write the same equation into we can write the above equation into

$$h'(t) = G(t)e^{-ct},$$

where

$$G(t) = \frac{e^{-(\theta-c)t - \frac{h(t)-\theta}{k}[1-e^{-kt}]} - 1}{\frac{h(t)}{c(h(t)-\theta)}[1-e^{-ct}] + \frac{1}{h(t)-\theta}\{e^{-\theta t - \frac{h(t)-\theta}{k}[1-e^{-kt}]} - 1\}}.$$

Since it has been shown that $\lim_{t\to\infty} h(t) = h^* < c$, it is straightforward to show that $G(t)$ is uniformly bounded in t and

$$\lim_{t\to\infty} G(t) = \frac{kc(h^* - \theta)}{h^* - c}.$$

Now we postulate

$$h(t) \sim h^* + \rho_2 e^{-ct}, \quad \text{if } c < \theta.$$

Compare the limit of $h'(t)$, we get

$$\rho_2 = \frac{k(h^* - \theta)}{h^* - c}.$$

Corollary 6.3. *As $t \to \infty$,*

$$h(t) \sim h^* - \rho_1 e^{-\theta t} + \rho_2 e^{-ct},$$

where h^*, ρ_1, and ρ_2 are defined in Theorem 3.

6.6 Global Approximation Formulas

We propose that the free boundary $h(t)$ globally behaves like

$$h(t) \sim h^* - (h^* - c)e^{-\beta t}, \tag{6.13}$$

where clearly $h \to h^*$ as $t \to \infty$, and β is chosen to match the asymptotic expansion of $h(t) \sim c + \alpha t$, which means

$$\beta = \frac{k(c - \theta)}{3(h^* - c)} \tag{6.14}$$

The accuracy of approximation can be improved if we use a little bit more complicated interpolation formula

$$h(t) \sim h^* - (h^* - c)\exp[1 - e^{\beta t}]. \tag{6.15}$$

We choose a super exponential function to make the free boundary "decay" faster to the true boundary, when other conditions are matched as previous. And also this does not alter the asymptotic expansion at t infinity at all. In the same rationale, β is chosen to match the asymptotic expansion of $h(t) \sim c + \alpha t$ for t small, which gives the same expression of β as defined in (6.14). In Fig. 6.2 we provide a comparison of our analytical approximations and the true numerical solution of the free boundary.

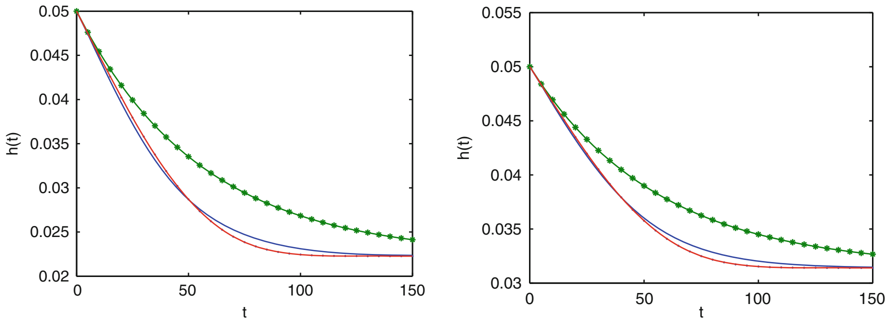

Fig. 6.2 The plain curve is the true solution. The top stared curve is the first approximation, and the bottom dotted curve is the second approximation. $c = 0.05, \theta = 0.06, k = 0.15$ (*left*), 0.10 (*right*). The units for t and $h(t)$ are years and year^{-1}, respectively

In general these approximation formulas are very accurate. Our numerical experiments with a variety of parameters show that the relative error is within 4% for $t < 20$ for the second formula. From the financial practitioner's point of view, both our numerical method and the approximation formula can provide satisfactory solutions.

6.7 Conclusion

Assuming the underlying interest rate follows the CIR model, we studied the mortgage borrower's optimal strategy to make prepayments when the volatility of market return rate is small. We derived the integral equation representation of the solution and studied the mathematical properties of the free boundary. An efficient iteration scheme was designed to solve the free boundary numerically. We also found two useful approximation formulas, the accuracy of which are validated with numerical simulations.

References

1. J. Cox, J. Ingersoll, & S. Ross, *A theory of the term structure of interest rates*, Econometrica , **53** (1985), 385–407.
2. Schwarts, E.S., & W.N. Torous, *Prepayment and the valuation of mortgage-backed securities*, J. of Finance **44**, 375–392.
3. D. Xie, X. Chen & J. Chadam, *Optimal Termination of Mortgages*, European Journal of Applied Mathematics, 2007.
4. L. Jiang & A. Rennie, *Mathematical Modeling and Methods of Option Pricing*, World Scientific Publishing House, 2005.
5. S.A. Buser, & P. H. Hendershott, *Pricing default-free fixed rate mortgages*, Housing Finance Rev. **3** (1984), 405–429.

6. D. Xie, *Parametric estimation for treasury bills*, International Research Journal of Financial Economics, 17 (2008), 27–32.
7. L. Jiang, B. Bian & F. Yi. *A parabolic variational inequality arising from the valuation of fixed rate mortgages*, European J. Appl. Math. **16** (2005), 361–338.
8. P. Willmott, *Derivatives, the theory and practice of financial engineering*, John Wiley & Sons, New York, 1999.
9. J. Hull, *Options, Futures and Other Derivatives* , Prentice Hall, 2005.
10. A. Etheridge, *A course in Financial Calculus*, Cambridge University Press, 2004.