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An Alternative Example of the Method of Multiple Scales*

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Abstract. An alternative example of the method of multiple scales is presented. This example arises in the study of the classical heat equation with a slowly varying flux imposed at one end. The module presents introductory ideas about dimensionless variables, multiple-scale expansions, and scaling of the dependent variable. The necessarily obfuscating algebraic computation is less than that for more familiar multiple-scale examples, such as the perturbed oscillator. The results are analyzed for both their physical and mathematical importance.

Key words. perturbation methods, multiple-scale expansions, education, heat equation, scaling

AMS subject classifications. 35-01, 35B25, 35C20, 35K05, 80A20

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1. Introductions.

1.1. To the Instructor.

In the context of a course in singular perturbation theory, the method of multiple scales is often presented. For historical reasons, the classical example of the method of multiple scales has been that of a perturbed harmonic oscillator [1], [2], [3], [4]. Though illustrative, this example (especially in the case of nonlinear problems) suffers from the complicated nature of the algebraic manipulations required for its solution. Rooting through all this algebra detracts from the new idea—the multiple-scale expansion—being presented.

In the context of studying a more complicated problem involving reactions on the surface of a DNA strip [5], the example presented herein was encountered. The algebra required is much reduced, and hence does not obfuscate the key ideas. In addition, the example introduces simply the concept of multiple-scale expansions for partial differential equations (PDEs). Other examples resort to traveling waves to reduce the problem to an ordinary differential equation (ODE) [2], [3]. We reduce the problem to an ODE using separation-of-variables techniques, which are applicable to a wider class of problems.

The module is designed to be used optimally by students who have had some exposure to both PDEs and perturbation methods. However, the initial separation-of-variables solution is derived in section 3 from first principles, so students without a background in PDEs can follow the analysis. In addition, the key ideas of perturbation methods are explicated herein, and thus it may be used by students with little or no background in that subject.

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1.2. To the Student. Many physical problems involve a balance of dynamical mechanisms. Sometimes all the mechanisms are important at all times and places, but often they are not. To see the balances mathematically, we often introduce *dimensionless variables* by dividing the variables by constant *characteristic values*.

For instance, when studying wave motion on the ocean, a "natural" length scale is the length of the boat. In a bacteria growth problem, a natural time scale is the time needed for the population to double [6]. In an artillery problem, a possible length scale (though not a very useful one) is the radius of the earth [7]. Since only one scale may be chosen for each variable, the balances between mechanisms appear as ratios between parameters.

Perhaps in one region or on one time scale two physical effects are most important. Then in that region, as a good first approximation we may assume that these are the *only* effects in the problem. Mathematically, the coefficients of terms in an equation corresponding to other physical processes may be quite small, and hence we hope that those terms can be neglected. The technique just described falls under the category of *perturbation methods*. In this paper we concentrate on the *method of multiple scales*, where the nature of the problem forces us to examine the solution on two or more scales at the same time.

In this module we demonstrate the method of multiple scales by examining the case of heat conduction (diffusion of heat) in a rod where a slowly varying heat flux is imposed at one end. We begin by writing down the dimensional governing equations for the system. By choosing appropriate scales associated with heat conduction, we derive a far simpler system containing a small parameter ϵ. As a first approximation, we ignore the variance of the heat flux. Unfortunately, this yields a divergent solution. Next we ignore the "fast" heat conduction. This yields an inconsistent system. The only way out is to consider *both* time scales simultaneously; hence the name "method of multiple scales."

2. Governing Equations. We wish to examine the problem of heat flow in a thin cylindrical rod of length L. (A table of all the variables used in this module, along with units and equation of first introduction, is given in section 6.) As a first approximation, we take the rod to be one-dimensional. (The enterprising reader may wish to show that this is a good approximation; see Exercise 3.1.) The governing equation for the heat flow is derived elsewhere [7], [8], [9], [10]; here we simply quote the result:

$$\rho\hat{C}_p\frac{\partial\tilde{\theta}}{\partial\tilde{t}} = -\frac{\partial\tilde{J}}{\partial\tilde{x}}, \qquad 0 \le \tilde{x} \le L, \tag{2.1a}$$

where ρ is the density of the rod, $\hat{C}_p$ is the heat capacity at constant pressure of the rod, $\tilde{J}$ is the heat flux, $\tilde{\theta}$ is the temperature, and $\tilde{x}$ measures distance along the rod.

Equation (2.1a) states that if the heat flux flowing into a segment $d\tilde{x}$ of the rod from the left is more than that flowing out to the right, then the temperature at $\tilde{x}$ will increase. The temperature will decrease if the converse is true. The heat flux $\tilde{J}$ is given by

$$\tilde{J} = -k\frac{\partial\tilde{\theta}}{\partial\tilde{x}}, \tag{2.1b}$$

where k is the thermal conductivity of the rod. Equation (2.1b) states that heat will flow from areas of high temperature to areas of low temperature, and that the relationship is linear. Equation (2.1b) is called *Fourier's law of heat conduction* [9]. Combining equations (2.1) yields the standard heat equation

$$\rho \hat{C}_p \frac{\partial \tilde{\theta}}{\partial \tilde{t}} = k \frac{\partial^2 \tilde{\theta}}{\partial \tilde{x}^2}, \qquad 0 \leq \tilde{x} \leq L. \tag{2.2}$$

Initially there is a uniform temperature of zero in the rod:

$$\tilde{\theta}(\tilde{x}, 0) = 0. \tag{2.3}$$

At the two ends of the rod, we specify the heat flux $\tilde{J}$. At the end $\tilde{x} = L$, we require that the rod be *insulated*; that is, there is no heat flux through this end:

$$\tilde{J}(L, \tilde{t}) = -k \frac{\partial \tilde{\theta}}{\partial \tilde{x}}(L, \tilde{t}) = 0. \tag{2.4a}$$

At the end $\tilde{x} = 0$, we specify the heat flux to be some known function. We wish to vary the heat flux slowly, so we associate a small frequency ω with the time in the function:

$$\tilde{J}(0, \tilde{t}) = -k \frac{\partial \tilde{\theta}}{\partial \tilde{x}}(0, \tilde{t}) = -J_{\mathrm{c}} f(\omega \tilde{t}), \tag{2.4b}$$

where J_{c} is a characteristic size of the heat flux and f is a dimensionless function. By "characteristic size" we simply mean a value that the flux neither exceeds by a large quantity nor is much smaller than. Candidates include the amplitude of an oscillatory function, the mean, or the root mean square.

Equations (2.1)–(2.4) have been written in *dimensional form*; that is, all the variables should be considered to have units associated with them. (The particular units may be found by consulting section 6.) However, often it is *relative* differences that are important. For instance, if during the experiment the bar is 25°C, is that "cold" or "hot"? The answer would most assuredly depend on whether the bar started the experiment at −25°C or 125°C. What do we mean by "varying the heat flux slowly"? More precisely, the variation should be slow compared with the characteristic time needed for a "heat signal" to diffuse across the bar. This way, any change in the heat flux can be accommodated by conduction in the bar.

In describing this characteristic time in terms of the physical problem, students of the heat equation may object that the diffusion operator allows an infinite signal speed, and thus some heat signal is transferred throughout the bar instantaneously [10]. However, this signal is infinitesimal, and hence we could choose a time at which a signal of half the original intensity passes to the end of the bar. Fortunately, we need not choose such an artificial time; the natural conduction time will be revealed to us shortly as a consequence of our scaling. Another insightful question to ask is, What is the size of this characteristic time? Clearly if the bar is very short or the thermal conductivity k is very large, it will be short, and it will be long if the opposite is true.

To resolve these issues, we wish to scale all variables in the problem to make them dimensionless. (Once we have an answer, it is a simple matter to reverse the process to express all quantities in dimensional form.) To do this, we must divide each variable by a characteristic value. The goal is to reduce the number of parameters if at all possible and to simplify any algebra we can. If we let

$$x = \frac{\tilde{x}}{L}, \tag{2.5a}$$

where x is now dimensionless, we reduce the domain to $x \in [0,1]$, the simplest possible domain. However, it is not immediately obvious how we should scale time and temperature, so we let

$$t = \omega_{\mathrm{c}} \tilde{t}, \qquad \theta(x,t) = \frac{\tilde{\theta}(\tilde{x},\tilde{t})}{\theta_{\mathrm{c}}}, \tag{2.5b}$$

where ω_{c} and θ_{c} are constants to be determined. Making the substitutions listed in (2.5) into (2.2), we obtain

$$\rho \hat{C}_p \theta_{\mathrm{c}} \omega_{\mathrm{c}} \frac{\partial \theta}{\partial t} = \frac{k\theta_{\mathrm{c}}}{L^2} \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 \le x \le 1,$$

$$\frac{\partial \theta}{\partial t} = \frac{k}{L^2 \rho \hat{C}_p \omega_{\mathrm{c}}} \frac{\partial^2 \theta}{\partial x^2}. \tag{2.6}$$

We are interested in the heat conduction in the rod, and thus wish to keep the coefficient on the right-hand side of a moderate size. That is, we do not wish the coefficient to be so large that the left-hand side may be ignored, nor so small that the right-hand side may be ignored. We strive for a balance between the two terms.

Clearly the easiest way to create such a balance (from both a physical and an algebraic standpoint) is to set the coefficient equal to 1. Therefore, we choose

$$\omega_{\mathrm{c}} = \frac{k}{\rho \hat{C}_p L^2}, \tag{2.7}$$

which makes (2.6)

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}. \tag{2.8}$$

The inverse of ω_{c} is exactly the natural conduction time discussed above. We note that if k is large or L is small, the frequency ω_{c} is very high, which corresponds to a short characteristic diffusion time, as suspected.

Substituting (2.5) and (2.7) into (2.4b), we obtain

$$\frac{k\theta_{\mathrm{c}}}{L} \frac{\partial \theta}{\partial x}(0,t) = J_{\mathrm{c}} f\left(\frac{\omega t}{\omega_{\mathrm{c}}}\right),$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{J_{\mathrm{c}} L}{k\theta_{\mathrm{c}}} f(\epsilon t), \qquad \epsilon = \frac{\omega}{\omega_{\mathrm{c}}}. \tag{2.9}$$

Note that ϵ is a dimensionless ratio of the frequency of the flux oscillations and the frequency associated with conduction. The form of ϵ allows us to quantify what we mean by "slow variance." We want the frequency of the imposed heat flux oscillation to be much less than the frequency due to diffusion, so we require that $0 < \epsilon \ll 1$. We want to be sure to include the effects of the flux forcing, and thus wish to keep the coefficient on the right-hand side of a moderate size. As above, we would ideally like the coefficient to be 1, so we choose

$$\theta_{\mathrm{c}} = \frac{J_{\mathrm{c}} L}{k}. \tag{2.10}$$

This makes (2.3), (2.4a), and (2.9)

$$\theta(x,0) = 0, \tag{2.11}$$

$$\frac{\partial\theta}{\partial x}(1,t) = 0, \tag{2.12a}$$

$$\frac{\partial\theta}{\partial x}(0,t) = f(\epsilon t). \tag{2.12b}$$

Note the vast simplification accomplished by introducing dimensionless variables. We have reduced the size of our parameter set from six (L, k, ρ, $\hat{C}_p$, J_c, and ω) down to just one: ϵ. Introducing dimensionless variables will always reduce the number of parameters, a result called the *Buckingham Pi theorem* [6]. In addition, it reinforces the relativity of actual physical measurements. No matter how small ω_c is, we simply require that ω be smaller to enforce that $\epsilon \ll 1$.

By reducing the number of parameters, we note that we can now solve a wide range of physical problems by constructing the solution to the system above, then substituting in the parameter values to determine ϵ and the dimensional quantities.

EXERCISE 2.1. *Clearly equations in dimensional quantities make sense only if the terms on each side of the equation have the same dimensions. That is,*

$$3\,\frac{\mathrm{m}}{\mathrm{sec}} \textit{ clearly does not equal } 3\,\frac{\mathrm{g}}{\mathrm{cm}^3}.$$

Using the dimensions listed in section 6 *for each of the dimensional quantities, verify that the terms in each of equations* (2.1) *and* (2.4b) *have the same dimensions.*

EXERCISE 2.2. *Verify that ω_c has units of inverse time and θ_c has units of temperature.*

EXERCISE 2.3. *shape Using appropriate resources (for instance,* [11], [12], [13], [14]*), estimate ω_c for rods of various materials. How slow should the flux oscillations be in these cases?*

3. Perturbation Methods. Since the system (2.8), (2.11), and (2.12) is linear, its solution may be written down via the technique of separation of variables [8], [10]. We define

$$\theta_n(t) = 2\int_0^1 \theta(x,t)\cos(n\pi x)\,dx, \qquad \theta(x,t) = \frac{\theta_0(t)}{2} + \sum_{n=1}^{\infty} \theta_n(t)\cos(n\pi x). \tag{3.1}$$

Here θ_n is sometimes called the *finite Fourier cosine transform* of θ [15]. If we can obtain an equation for θ_n alone, it will be an ordinary differential equation (ODE), rather than a partial differential equation (PDE). Then upon solving for θ_n for each n, we may substitute our result into the sum in (3.1) to obtain the solution $\theta(x,t)$.

To obtain an ODE for θ_n, we multiply (2.8) by $2\cos(n\pi x)$ and integrate from $x = 0$ to $x = 1$. Integrating by parts twice, we obtain

(3.2a)

$$\int_0^1 2\frac{\partial\theta}{\partial t}\cos(n\pi x)\,dx = \int_0^1 2\frac{\partial^2\theta}{\partial x^2}\cos(n\pi x)\,dx,$$

$$2\frac{\partial}{\partial t}\int_0^1 \theta\cos(n\pi x)\,dx = 2\left[\frac{\partial\theta}{\partial x}\cos(n\pi x)\right]_0^1 + 2n\pi\int_0^1 \frac{\partial\theta}{\partial x}\sin(n\pi x)\,dx$$

$$= -2\frac{\partial\theta}{\partial x}(0) + 2n\pi\left[\theta\sin(n\pi x)\right]_0^1 - n^2\pi^2\left[2\int_0^1 \theta\cos(n\pi x)\,dx\right],$$

$$\frac{d\theta_n}{dt} = -2f(\epsilon t) - n^2\pi^2\theta_n, \tag{3.2b}$$

where we have used (2.12).

The choice of the cosine series in (3.1) was not arbitrary; it was chosen so that we could use the boundary data given in (2.12) to reduce (3.2b) to an ODE where all quantities except θ_n are known. It is the *only* series for which the problem simplifies so completely. For instance, a sine series will not work; see Exercise 3.2.

The initial condition for (3.2b) is given by the finite Fourier cosine transform of (2.11):

$$\theta_n(0) = 0. \tag{3.3}$$

Solving (3.2b) subject to (3.3), we have the following:

$$\frac{d}{dt}\left(e^{n^2\pi^2 t}\theta_n\right) = -2f(\epsilon t)e^{n^2\pi^2 t}\theta_n,$$

$$\theta_n(t) = -2\int_0^t e^{-n^2\pi^2(t-z)}f(\epsilon z)\,dz. \tag{3.4}$$

This expression, though accurate, may not be very useful in computing and interpreting our solutions, especially if f is a complicated function. In particular, we may not be able to evaluate the integral in closed form.

Substituting (3.4) into (3.1), we note that the resulting series is useful for large time, since the terms $e^{-n^2\pi^2 t}$ decay exponentially fast with increasing n. Thus summing only a few terms can lead to an excellent approximation. Unfortunately, for small times the terms decay very slowly, and hence the series solution is not very useful.

What we desire to do is exploit the assumed smallness of ϵ to see if we can simplify our work while obtaining a more convenient expression. Physically, we expect that the oscillation in the heat flux will be so slow that the rod can conduct any resulting changes in the temperature away, thus equilibrating the temperature. Therefore, we expect the rod to be near the equilibrium state for all time. Mathematically, we wish to use a perturbation method [1], [2], [3], [4], [16] to obtain an approximate solution for θ_n. Since ϵ is so small, we expand all of our relevant functions in Taylor series in ϵ:

$$\theta_n(t;\epsilon) = \theta_n^0(t) + \epsilon\theta_n^1(t) + \cdots, \qquad f(\epsilon t) = f(0) + \epsilon t f'(0) + \cdots. \tag{3.5}$$

In order for the series to remain valid, we require that each term in the expansion be smaller than the one before. (Otherwise, we would have no assurance that this approximation accurately represents the true solution.) We substitute (3.5) into (3.2b) and (3.3) and equate like powers of ϵ:

$$\frac{d(\theta_n^0 + \epsilon\theta_n^1)}{dt} = -2[f(0) + \epsilon t f'(0)] - n^2\pi^2(\theta_n^0 + \epsilon\theta_n^1),$$

$$\frac{d\theta_n^0}{dt} + n^2\pi^2\theta_n^0 = -2f(0), \tag{3.6a}$$

$$\frac{d\theta_n^1}{dt} + n^2\pi^2\theta_n^1 = -2tf'(0), \tag{3.6b}$$

$$(\theta_n^0 + \epsilon\theta_n^1)(0) = 0,$$

$$\theta_n^0(0) = 0, \tag{3.7a}$$

$$\theta_n^1(0) = 0. \tag{3.7b}$$

The solution of (3.6) and (3.7) proceeds just as in the solution of (3.2b) and (3.3), but the forcing functions are now so simple that we may obtain closed-form expressions. At this stage we focus only on the mode $n = 0$, which represents twice the average temperature $\bar{\theta}$ in the bar, since

$$\theta_0(t) = 2\int_0^1 \theta(x,t)\,dx = 2\bar{\theta}.$$

For $n = 0$, the equation analogous to (3.4) is

$$\theta_0^0(t) = -2\int_0^t f(0)\,dz = -2f(0)t. \tag{3.8a}$$

Solving for θ_0^1 in the same way, we obtain the following:

$$\theta_0^1(t) = -2\int_0^t zf'(0)\,dz = -f'(0)t^2. \tag{3.8b}$$

If we use the series form for θ_0 given in (3.5) and our expressions (3.8), we see that to leading two orders we have

$$\theta_0(t) \sim -2f(0)t - \epsilon f'(0)t^2. \tag{3.9}$$

We note that the expression (3.9) is exactly what we would have obtained had we substituted our series form for $f(t)$ given in (3.5) into (3.4) with $n = 0$:

$$\theta_0(t) \sim -2\int_0^t f(0) + \epsilon z f'(0)\,dz = -2f(0)t - \epsilon f'(0)t^2.$$

The expression (3.9) is a good approximation for $t \ll \epsilon^{-1}$, which (depending on the size of ϵ) may be good enough for the experiment at hand. In addition, (3.9) yields a closed-form expression that may be easily calculated for any f, as opposed to (3.4), which does not.

However, we note that as t grows large (in particular, for $t \gg \epsilon^{-1}$), the second term is larger than the first, which violates the assumption of the perturbation expansion given after (3.5). (A similar phenomenon happens for $n \neq 0$; see Exercise 3.3.) This latter term that grows too large is called a *secular term* [2], [3]. We also note that the first term diverges as $t \to \infty$. This may or may not be a problem, depending on the form of f.

This paradox illustrates the contradictory nature of the asymptotic expansion. Note that as $\epsilon \to 0$ for any *finite* t, the series in (3.9) converges. However, for any *finite* ϵ, the series diverges as $t \to \infty$. We say that the series *loses validity* whenever $t \gg \epsilon^{-1}$. Why this series failed can be seen by letting $f(\epsilon t) = \cos(\epsilon t) - \sin(\epsilon t)$. Then the small-$\epsilon$ expansion

$$\cos(\epsilon t) - \sin(\epsilon t) = 1 - \epsilon t + \cdots$$

clearly does not hold for t large. In particular, for $t \gg \epsilon^{-1}$, the series above diverges, while $\cos(\epsilon t) - \sin(\epsilon t)$ is always bounded. This phenomenon is illustrated in Figure 3.1.

Therefore, we see that we must treat variations in the heat flux more carefully than just expanding them in a Taylor series. As a first attempt at resolving the

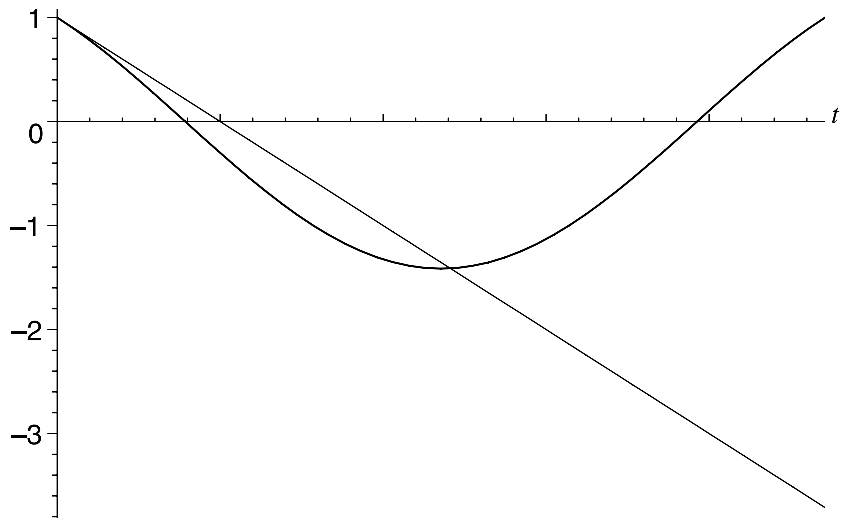

Fig. 3.1 $1 - \epsilon t$ *(thin line) and* $\cos(\epsilon t) - \sin(\epsilon t)$ *(thick line) vs.* t *for* $\epsilon = 0.01$.

problem, we rescale time so that the oscillations in the heat flux occur on the time scale of interest. These oscillations occur on the time scale ϵt, so we let

$$\tau = \epsilon t, \qquad \phi(x,\tau) = \theta(x,t). \tag{3.10}$$

(In contrast to rescaling t, one can find the scale τ directly from the dimensional equations by choosing ω_{c} differently; see Exercise 3.4.)

Substituting (3.10) into (2.8), (2.11), and (2.12), we obtain

$$\epsilon \frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial x^2}, \tag{3.11}$$

$$\phi(x,0) = 0, \tag{3.12}$$

$$\frac{\partial \phi}{\partial x}(1,\tau) = 0, \tag{3.13a}$$

$$\frac{\partial \phi}{\partial x}(0,\tau) = f(\tau). \tag{3.13b}$$

Again we assume a perturbation series in ϵ for our dependent variable (in this case, ϕ):

$$\phi(x,\tau;\epsilon) = \phi^0(x,\tau) + \epsilon \phi^1(x,t) + \cdots. \tag{3.14}$$

Substituting (3.14) into (3.11) and (3.13), we have, to leading order,

$$\frac{\partial^2 \phi^0}{\partial x^2} = 0, \tag{3.15}$$

$$\frac{\partial \phi^0}{\partial x}(1,\tau) = 0, \tag{3.16a}$$

$$\frac{\partial \phi^0}{\partial x}(0,\tau) = f(\tau). \tag{3.16b}$$

Note that the leading-order operator (3.15) does not allow an initial condition to be imposed upon it, and thus we didn't bother substituting (3.14) into (3.12). This type of behavior often leads to a *singular perturbation problem*; more details about the elimination of boundary conditions may be found in [2], [3].

Solving (3.15), we obtain

$$\begin{aligned} \phi(x,\tau) &= A(\tau)x + B(\tau) \\ &= B(\tau), \end{aligned} \tag{3.17}$$

where we have used (3.16a). But we note that (3.17) does not solve (3.16b), and hence there is no solution to the system (3.15) and (3.16). This is due to the fact that the leading-order operator, the one-dimensional Laplace's equation, is a *steady-state* operator; i.e., it does not have any explicit time dependence. But we are trying to impose a time-dependent flux at one end. Physically, the conduction mechanism must equilibrate any changes in the flux that occur. However, (3.15) models only steady-state phenomena, and hence the equilibration process cannot be described by this equation.

This contradiction may also be seen by looking at the finite Fourier cosine transform of ϕ. After using (3.10), (3.2b) becomes

$$\epsilon \frac{d\phi_n}{d\tau} = -f(\tau) - n^2\pi^2\phi_n. \tag{3.18}$$

Expanding ϕ_n in a perturbation series in ϵ à la (3.5) and substituting the results into (3.18), the equation corresponding to the leading order in ϵ is

$$f = -n^2\pi^2\phi_n^0.$$

But this implies that $f(\tau) = 0$ from the zeroth-mode ($n = 0$) equation, which doesn't make sense. When faced with such a contradiction, we must conclude that the form for our perturbation expansion is incorrect. (Note this is the only assumption we made; everything else followed directly from it.) Therefore, we were wrong to solve the problem on the diffusion time scale and expand our function $f(\epsilon t)$ as a series in ϵ, and we were wrong to solve the problem on the τ time scale and neglect conduction altogether. We must consider both time scales simultaneously.

Exercise 3.1. *Suppose that the cylindrical rod has some finite radius R. Write the two-dimensional equations that result. Using the perturbation methods outlined in this section, derive a relationship between R and L such that the leading-order equation is still the one-dimensional equation* (2.8).

Exercise 3.2. *Suppose that we replace the cosine by a sine everywhere in* (3.1) *and* (3.2a). *Show that upon integration by parts, the equation analogous to* (3.2b) *includes unknown boundary data and hence cannot be solved explicitly.*

Exercise 3.3. *Solve* (3.6) *and* (3.7) *for $n \neq 0$. Show that $\theta_n^1 \gg \theta_n^0$ for $t \gg \epsilon^{-1}$.*

Exercise 3.4. *Verify directly that by letting $\omega_c = \omega$ in the dimensional equations* (2.2)–(2.4), *we obtain* (3.11)–(3.13).

4. Multiple-Scale Expansion. Since the two time variables t and τ represent the time scales for two different but interdependent physical processes, it seems reasonable that trying to solve the system using one time scale alone would fail. Therefore, we include both time scales in our expansion by letting

$$\theta(x,t) = \Theta(x,T,\tau), \qquad T = t, \tag{4.1}$$

where we use the letter T for the fast time scale (corresponding to the diffusion process) to ease somewhat the confusion when we introduce the following transformation from the chain rule:

$$\frac{d}{dt} = \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau}. \tag{4.2}$$

Essentially we are assuming that T and τ are independent variables and we are ignoring their relationship to the original variable t. Substituting (4.1) and (4.2) into (2.8), (2.11), and (2.12), we obtain

$$\frac{\partial \Theta}{\partial T} + \epsilon \frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial x^2}, \tag{4.3}$$

$$\Theta(x, 0, 0) = 0, \tag{4.4}$$

$$\frac{\partial \Theta}{\partial x}(1, T, \tau) = 0, \tag{4.5a}$$

$$\frac{\partial \Theta}{\partial x}(0, T, \tau) = f(\tau). \tag{4.5b}$$

Therefore, we see that we have resolved several problematic issues from the previous section:

(1) We have explicitly retained the oscillation time τ in (4.5b), so we avoid expanding f in a series in ϵt as in (3.5). (Here we assume that f itself has no expansion in ϵ. This assumption can be relaxed; see Exercise 4.1.)

(2) We have explicitly retained the conduction time T in (4.3), so we will not have an inconsistent system as in (3.15).

(3) In (3.4), we note that the variable t (through the variable z) appears in two ways. It appears with ϵ in the argument of f, and alone in the exponential. By using T and τ, we are no longer approximating either of these forms; we are using both of them explicitly.

We again use a cosine series expansion as in section 3. Thus the analogous equations to (3.2b) and (3.3) are

$$\frac{\partial \Theta_n}{\partial T} + \epsilon \frac{\partial \Theta_n}{\partial T} + n^2\pi^2\Theta_n = -2f(\tau), \tag{4.6a}$$

$$\Theta_n(0, 0) = 0. \tag{4.6b}$$

We assume a Taylor series for Θ_n in ϵ:

$$\Theta_n(T, \tau; \epsilon) = \Theta_n^0(T, \tau) + \epsilon\Theta_n^1(T, \tau) + \cdots. \tag{4.7}$$

Substituting (4.7) into (4.6), we obtain, to leading two orders,

$$\frac{\partial(\Theta_n^0 + \epsilon\Theta_n^1)}{\partial T} + \epsilon \frac{\partial(\Theta_n^0 + \epsilon\Theta_n^1)}{\partial T} + n^2\pi^2(\Theta_n^0 + \epsilon\Theta_n^1) = -2f(\tau),$$

$$\frac{\partial \Theta_n^0}{\partial T} + n^2\pi^2\Theta_n^0 = -2f(\tau), \tag{4.8a}$$

$$\frac{\partial \Theta_n^0}{\partial \tau} + \frac{\partial \Theta_n^1}{\partial T} + n^2\pi^2\Theta_n^1 = 0, \tag{4.8b}$$

$$(\Theta_n^0 + \epsilon\Theta^1)(0, 0) = 0,$$

$$\Theta_n^0(0, 0) = 0, \tag{4.9a}$$

$$\Theta_n^1(0, 0) = 0. \tag{4.9b}$$

Solving (4.8a), we have, for $n \neq 0$,

$$\Theta_n^0(T,\tau) = -\frac{2f(\tau)}{n^2\pi^2} + g_n(\tau)e^{-n^2\pi^2 T}, \tag{4.10a}$$

where

$$g_n(0) = \frac{2f(0)}{n^2\pi^2} \tag{4.10b}$$

from (4.9a). Note that at this stage the g_n are arbitrary. We must continue on to the next order to solve for them. This is due to the fact that in the expansion of θ_n for arbitrary n, the term that violates the expansion does not appear until one solves for θ_n^1 (see Exercise 3.3).

Substituting (4.10a) into (4.8b), we obtain the following:

$$\frac{\partial \Theta_n^1}{\partial T} + n^2\pi^2\Theta_n^1 - \frac{2f'(\tau)}{n^2\pi^2} + g_n'(\tau)e^{-n^2\pi^2 T} = 0,$$

$$\frac{\partial (e^{n^2\pi^2 T}\Theta_n^1)}{\partial T} = -n^2\pi^2 g_n'(\tau) + \frac{2f'(\tau)e^{n^2\pi^2 T}}{n^2\pi^2},$$

$$\Theta_n^1(T,\tau) = -n^2\pi^2 g_n'(\tau)Te^{-n^2\pi^2 T} + \frac{2f'(\tau)}{n^4\pi^4} + h_n(\tau)e^{-n^2\pi^2 T}. \tag{4.11}$$

Therefore, we see that if $g_n'(\tau) \neq 0$, then we have an undesirably large secular term as in (3.9). We *suppress* the secularity by setting

$$g_n'(\tau) = 0 \qquad \Longrightarrow \qquad g_n(\tau) = \frac{2f(0)}{n^2\pi^2},$$

$$\Theta_n^0(T,\tau) = \frac{2}{n^2\pi^2}\left[f(0)e^{-n^2\pi^2 T} - f(\tau)\right], \tag{4.12}$$

where we have used (4.10b).

Now we turn our attention to the case where $n = 0$. Then solving (4.8a) subject to (4.9a), we obtain

$$\Theta_0^0(T,\tau) = -2Tf(\tau) + g_0(\tau), \qquad g_0(0) = 0, \tag{4.13}$$

where the initial condition arises from satisfying (4.9a). Substituting (4.13) into (4.8b) yields the following:

$$-2Tf'(\tau) + g_0'(\tau) + \frac{\partial \Theta_0^1}{\partial T} = 0,$$

$$\frac{\partial \Theta_0^1}{\partial T} = 2Tf'(\tau) - g_0'(\tau),$$

$$\Theta_0^1(T,\tau) = T^2 f'(\tau) - g_0'(\tau)T + h_0(\tau), \tag{4.14}$$

and we again have the same problem as in (3.9). From the form of (4.13) and (4.14), it looks as if we are again trying to expand Θ_0^1 in a Taylor series which is not uniformly valid.

To ascertain what is going on, we examine the special case of $f(\epsilon t) = \sin \epsilon t$. Then calculating (3.4) for $n = 0$, we have

$$\theta_0(t) = -2\int_0^t \sin(\epsilon z)\, dz = \frac{2[\cos(\epsilon t) - 1]}{\epsilon}.$$

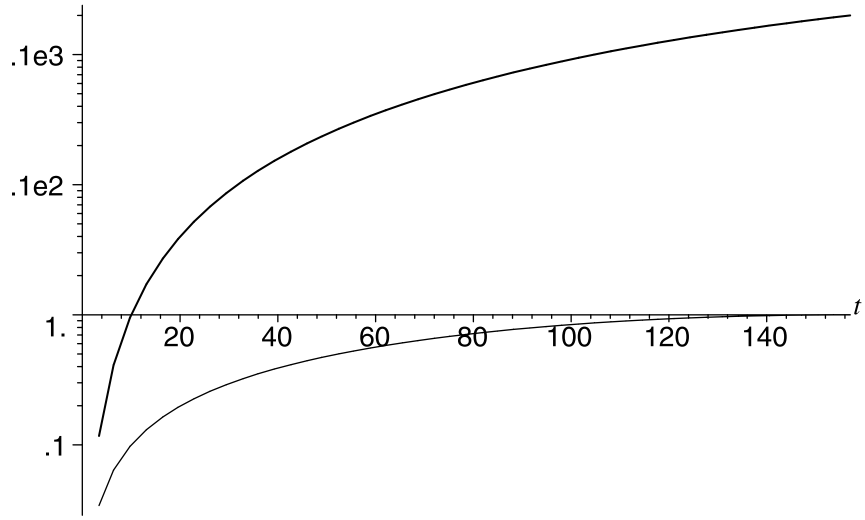

Fig. 4.1 *A lin-log plot of $f(t)$ (thin line) and $|\theta_0(t)|$ (thick line) vs. t for $\epsilon = 0.01$.*

Therefore, we see that θ_0 is really proportional to ϵ^{-1}, as can be seen in Figure 4.1, which shows $f(t)$ and $|\theta_0(t)|$ on the same axis. We note that since $f(t)$ varies so slowly, the contributions add up greatly, causing $\theta_0(t)$ to be quite large for even moderate t.

Thus our series assumptions in (3.5), (3.14), and (4.7) were incorrect, because they assumed that the first term in the expansion was proportional to ϵ^0. To remedy this, we modify (4.7) to be of the following form:

$$\Theta_n(T,\tau;\epsilon) = \epsilon^{-1}\Theta_n^{-1}(T,\tau) + \Theta_n^0(T,\tau) + \epsilon\Theta_n^1(T,\tau) + \cdots. \tag{4.15}$$

Substituting (4.15) into (4.6), we obtain, to leading three orders in ϵ,

$$\frac{\partial \Theta_n^{-1}}{\partial T} + n^2\pi^2\Theta_n^{-1} = 0, \tag{4.16a}$$

$$\frac{\partial \Theta_n^{-1}}{\partial \tau} + \frac{\partial \Theta_n^{0}}{\partial T} + n^2\pi^2\Theta_n^{0} = -2f(\tau), \tag{4.16b}$$

$$\frac{\partial \Theta_n^{0}}{\partial \tau} + \frac{\partial \Theta_n^{1}}{\partial T} + n^2\pi^2\Theta_n^{1} = 0, \tag{4.16c}$$

$$\Theta_n^{-1}(x,0,0) = 0, \tag{4.17a}$$

$$\Theta_n^{0}(x,0,0) = 0, \tag{4.17b}$$

$$\Theta_n^{1}(x,0,0) = 0. \tag{4.17c}$$

The solution to (4.16a) and (4.17a) is

$$\Theta_n^{-1}(T,\tau) = \Phi_n(\tau)e^{-n^2\pi^2 T}, \qquad \Phi_n(0) = 0, \tag{4.18}$$

where Φ_n is arbitrary at this stage for the same reasons that g_n was arbitrary before. Substituting (4.18) into (4.16b), we obtain

$$\Phi_n'(\tau)e^{-n^2\pi^2 T} + \frac{\partial \Theta_n^{0}}{\partial T} + n^2\pi^2\Theta_n^{0} = -2f(\tau),$$

$$\frac{\partial \Theta_n^{0}}{\partial T} + n^2\pi^2\Theta_n^{0} = -2f(\tau) - \Phi_n'(\tau)e^{-n^2\pi^2 T}. \tag{4.19}$$

We note that the second term on the right-hand side of (4.19) is proportional to the solution of the homogeneous operator on the left-hand side of (4.19). This phenomenon, called *forcing at resonance*, is related to the *Fredholm alternative theorem* [1], [17].

Solving (4.19) subject to (4.17b) for $n \neq 0$ yields the following:

$$\frac{\partial(e^{n^2\pi^2 T}\Theta_n^0)}{\partial T} = -\Phi_n'(\tau) - 2f(\tau)e^{n^2\pi^2 T},$$

$$\Theta_n^0(T,\tau) = -\Phi_n'(\tau)Te^{-n^2\pi^2 T} - \frac{2f(\tau)}{n^2\pi^2} + g_n(\tau)e^{-n^2\pi^2 T}, \tag{4.20a}$$

$$g_n(0) = \frac{2f(0)}{n^2\pi^2}, \tag{4.20b}$$

where the last equality comes from (4.17b). Note that if $\Phi_n'(\tau) \neq 0$, then we have the undesirable secularity as in (3.8b), because for T very large, $\Theta_n^0(T,\tau) \gg \Theta_n^{-1}(T,\tau)$. Therefore, we *suppress* the secularity by setting

$$\Phi_n'(\tau) = 0 \qquad \Longrightarrow \qquad \Phi_n(\tau) = 0,$$

$$\Theta_n^{-1}(T,\tau) = 0. \tag{4.21}$$

This corresponds to the fact that if we calculate θ_n for $n \neq 0$, we do not get a quantity that is proportional to ϵ^{-1} (see Exercise 4.3); only θ_0 is that large. We also note that upon substitution of (4.21) into (4.20a), we obtain our previous result (4.10a), and thus our previous result (4.12) holds as well.

For the case $n = 0$, (4.19) becomes

$$\frac{\partial\Theta_0^0}{\partial T} = -2f(\tau) - \Phi_0'(\tau),$$

$$\Theta_0^0 = -[2f(\tau) + \Phi_0'(\tau)]T + h_0(\tau), \tag{4.22}$$

and hence to suppress secularity we must let

$$\Phi_0(\tau) = -2F(\tau), \qquad F'(\tau) = f(\tau), \qquad F(0) = 0. \tag{4.23}$$

Making these substitutions into (4.18) with $n = 0$, we obtain

$$\Theta_0^{-1}(T,\tau) = -2F(\tau). \tag{4.24}$$

Note that if we substitute $\tau = \epsilon T$ into (4.24) and expand for small ϵ, we obtain the secular parts of (4.13) and (4.14). This is the only nonzero term in the expansion for Θ_0 (see Exercise 4.2).

EXERCISE 4.1. *Suppose that $f(\tau)$ has the following expansion:*

$$f(\tau) = f^0(\tau) + \epsilon f^1(\tau) + \cdots.$$

Write the equations for Θ_n^k in this case for $k = -1$, 0, and 1. Explain how your solution will change. (You need not work through all the details.)

EXERCISE 4.2. *Show that $\Theta_0^k = 0$ for any integer $k \geq 0$.*

EXERCISE 4.3. *Calculate θ_n from* (3.4) *for the case $f(\epsilon t) = \sin \epsilon t$, $n \neq 0$. Verify that this quantity is proportional to ϵ^0 for small ϵ.*

5. Conclusions.

5.1. Physical Interpretations. At this point we review the physical implications of our solution. By assumption, the oscillations in the heat flux occurred on a much slower time scale than that associated with heat conduction. Therefore, the bar equilibrated any change in the heat flux very quickly.

We began by focusing on the conduction time scale. On this time scale, the oscillations of the heat flux were so slow that the flux seemed to be nearly constant. This observation led us to expand $f(\epsilon t)$ in a series in ϵ, which turned out to be equivalent to expanding in a series in ϵt (see (3.5)). The nearly constant flux, integrated over a long time frame, produces a very large change in temperature, as shown in Figure 4.1. Unfortunately, our series solution was valid only for $\epsilon t \ll 1$.

Then we rescaled to focus on the oscillation time scale. On this time scale, the conduction is so fast that the bar immmediately reaches the steady state. But the steady-state equation cannot allow variation of the flux on *any* time scale (as shown in (3.15)), and hence we obtained an inconsistent system.

The solution to the problem involved incorporating *both* the oscillation and conduction time scales. Doing so, we obtained a solution that was uniformly valid for all time. As shown in (4.12), after a decay of the *initial* forcing on the conduction time scale, the solution behaves as if $f(\tau)$ were a constant. This is exactly what we would expect, since the oscillations are so slow that the forcing seems to be constant on the fast conduction time scale. However, this quasi-stationary forcing, integrated over the conduction time scale, causes a large buildup in mean temperature. Thus we found that our mean temperature, represented by θ_0, was proportional to ϵ^{-1}. Also, we noted that only θ_0 was that large; the other terms in the cosine series were smaller.

5.2. Mathematical Interpretations. When working with physical problems, it is convenient to introduce dimensionless variables by dividing the dimensional variables by characteristic scales. Not only does doing so reduce the number of parameters in the problem by the Buckingham Pi theorem [6], but also those parameters that remain provide insight into the physical processes at hand. In particular, they are the ratios of scales associated with various dynamical processes (in our case, imposed flux and conduction).

If two of the processes occur on widely disparate time scales, one of these ratios (call it ϵ) may become exceedingly small. In that case, one can write all the dependent variables as series in ϵ. By equating like powers of ϵ, one obtains a simpler set of equations to solve, and the solutions thus obtained are usually easier to analyze, as in (3.9).

However, such a regular perturbation solution may not be valid everywhere, causing the problem to be of a singular nature. The solution may fail to hold near a boundary. This often arises if a boundary condition cannot be satisfied; in our example, (3.15) could not satisfy the initial condition. In this case a *boundary layer* may need to be inserted: consult [1], [2], [3], [4], and [16] for more details.

Alternatively, the solution may not hold for all time. This usually is indicated by the appearance of a secular term which for some range of t is larger than the term before it in the expansion. An interesting aside: Note that if we had not carried our expansion to the next order, we would not have discovered the secularity. Therefore, though highly unlikely, it is always possible that a secular term lurks in some "lower order" term in the expansion, no matter how many terms we calculate or how carefully we construct our expansion.

If a secular term appears, a multiple-scale expansion technique is advisable. In this case, one treats the solution as if the variables T and τ were *independent*, rather than related through the variable t. By introducing the slow-time variable τ, one can then suppress any secularities by solving appropriate equations for the functions of τ.

The theory of multiple-scale expansions has been applied extensively to perturbed oscillators; the interested student can find canonical examples in [1], [2], [3], [4].

6. Nomenclature.

6.1. Variables and Parameters. Units are listed in terms of length (L), mass (M), temperature (Θ), or time (T). If the same letter appears both with and without tildes, the letter with a tilde has dimensions, while the letter without a tilde is dimensionless. The equation where a quantity first appears is listed, if appropriate.

$A(\tau)$:	arbitrary function of integration
$B(\tau)$:	arbitrary function of integration
$\hat{C}_p$:	heat capacity of the rod at constant pressure, units $L^2T^{-2}\Theta^{-1}$ (2.1a)
$F(\tau)$:	function in multiple-scale expansion, defined by $F'(\tau) = f(\tau)$ (4.23)
$f(\cdot)$:	dimensionless function describing imposed flux oscillations (2.4b)
$g_n(\tau)$:	arbitrary function in multiple-scale expansion (4.10a)
$h_n(\tau)$:	arbitrary function in multiple-scale expansion (4.11a)
$\tilde{J}(\tilde{x}, \tilde{t})$:	heat flux, units MT^{-3} (2.1a)
k:	thermal conductivity, units $MLT^{-3}\Theta^{-1}$ (2.1b)
L:	length of rod, units L (2.1a)
n:	indexing variable (3.1)
T:	fast time variable, value t (4.1)
$\tilde{t}$:	dimensional time, units T (2.1a)
$\tilde{x}$:	dimensional length, units L (2.1a)
$\mathcal{Z}$:	the integers
z:	dummy variable (3.4)
ϵ:	ratio of frequencies, considered to be small, value ω/ω_c (2.9)
$\Theta(x, T, \tau)$:	temperature incorporating both scales (4.1)
$\tilde{\theta}(\tilde{x}, \tilde{t})$:	dimensional temperature, units Θ (2.1a)
ρ:	density of the rod, units ML^{-3} (2.1a)
τ:	slow time variable, value ϵt (3.10)
$\Phi_n(\tau)$:	arbitrary function in multiple-scale expansion (4.18)
$\phi(x, \tau)$:	temperature in slow-time coordinates (3.10)
ω:	frequency of flux oscillations, units T^{-1} (2.4b)

6.2. Other Notation.

c:	as a subscript, used to indicate a characteristic value (2.4b)
$n \in \mathcal{Z}$:	as a subscript, used to indicate a term in a cosine expansion (3.1); as a superscript, used to indicate a term in a perturbation expansion in ϵ (3.5)
$\bar{}$:	used to denote the mean of the temperature

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