WKB Connection Formulas (Revised)

Consider the canonical WKB simple turning-point problem

$$\frac{d}{d\xi}\left(p(\xi)\frac{dy}{d\xi}\right)+[k^2q(\xi)-r(\xi)]y=0, \qquad k>0, \qquad p(\xi)>0, \tag{1a}$$

$$P(0)=P, \qquad q(\xi)\sim Q\xi, \quad \xi\to 0, \qquad Q<0. \tag{1b}$$

Then the outer solutions are given by

$$y_-=\frac{ce^z+de^{-z}}{|pq|^{1/4}}, \qquad \xi>0, \tag{2a}$$

$$y_+=\frac{1}{(pq)^{1/4}}\left[a\cos\left(z+\frac{\pi}{4}\right)+b\sin\left(z+\frac{\pi}{4}\right)\right], \qquad \xi<0, \tag{2b}$$

$$z=k\int_0^{\xi}\sqrt{\left|\frac{q(t)}{p(t)}\right|}\,dt. \tag{3}$$

Here the subscript on y refers to the sign of $q(\xi)$, not ξ. Also note that we define z such that $\operatorname{sgn}(z)=\operatorname{sgn}(\xi)$.

The connection formulas for this case are

$$a=2d, \qquad b=c. \tag{4}$$

If $Q>0$, locally we may map $\xi\mapsto-\xi$, so $z\mapsto -z$ and the connection formulas don't change. In summary, we have

$$\begin{array}{ccc} \text{oscillatory } (\xi<0) & \to & \text{exponential } (\xi>0) \\ \text{coefficient of } \sin(z+\pi/4) & \Longleftrightarrow & \text{coefficient of } e^z \\ \text{coefficient of } \cos(z+\pi/4) & \Longleftrightarrow & 2\times\text{coefficient of } e^{-z} \end{array} \tag{5$-$}$$

$$\begin{array}{ccc} \text{exponential } (\xi<0) & \to & \text{oscillatory } (\xi>0) \\ \text{coefficient of } e^{-z} & \Longleftrightarrow & \text{coefficient of } \sin(-z+\pi/4) \\ 2\times\text{coefficient of } e^{z} & \Longleftrightarrow & \text{coefficient of } \cos(-z+\pi/4) \end{array} \tag{5+}$$

WARNING: These connection formulas are good **ONLY** for the expressions in (1)–(3). Some authors put a negative sign between the two terms in (1), some use complex exponentials or two cosines with phase shifts instead of (2b), and other authors define a new variable $z_-=-z$ for $\xi<0$.

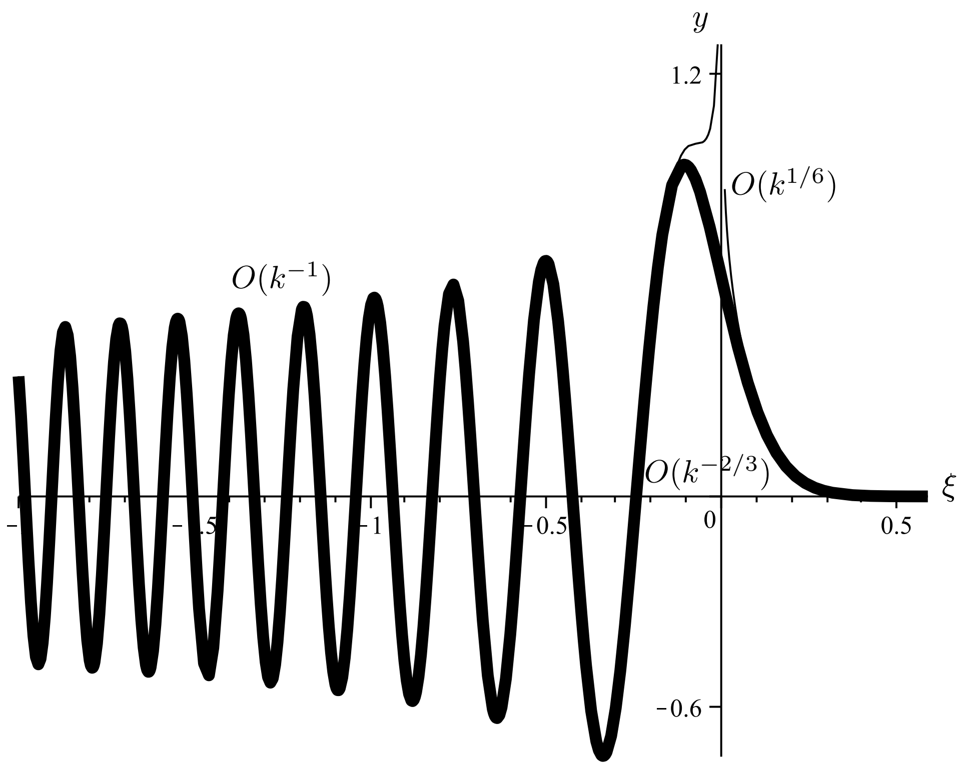
Consider the case where $p(\xi) = 1$, $q(\xi) = -\xi$, $r(\xi) = 0$. Then (1a) becomes

$$\frac{d^2y}{d\xi^2} - k^2\xi y = 0. \tag{5}$$

Considering only the bounded solution yields $c = 0$ in (2a). which forces $b = 0$ from (4) and $\beta = 0$ from notes in class. Hence the solutions are multiples of

$$\begin{aligned}
y_-(\xi) &= \frac{1}{2\xi^{1/4}\sqrt{\pi}} \exp\left(-\frac{2k\xi^{3/2}}{3}\right), \qquad \xi > 0, \\
y_+(\xi) &= \frac{1}{\xi^{1/4}\sqrt{\pi}} \sin\left(\frac{2k|\xi|^{3/2}}{3} + \frac{\pi}{4}\right), \qquad \xi < 0, \\
g(\xi) &= k^{1/6}\,\mathrm{Ai}(k^{2/3}\xi).
\end{aligned}$$

Note that since (5) is the Airy equation, $g(\xi)$ is the exact solution.



Here is a graph of the solution with $k = 20$. The thick line is g, while the thin lines are the outer solutions. Note that they diverge as $\xi \to 0$, while the exact solution only gets large there. Also note that the inner layer is wider than the oscillations of the exact solution when $\xi < 0$.