Homework Set 6 Solutions

1. (31 points) For the following problem:

\[ \ddot{u} + u - \epsilon u^2 = \epsilon \cos t, \quad 0 < \epsilon \ll 1, \]  \hspace{1cm} (6.1)

calculate the following:
(a) the proper expansion for \( u(t; \epsilon) \),
(b) the proper slow-time scale \( \tau \), and
(c) the proper evolution equations for \( A_0(\tau) \) and \( B_0(\tau) \).

You should also use Mathematica or Maple to graph the phase plane of \( A_0 \) and \( B_0 \).

**Solution.** The key to this problem is to balance the effects of the nonlinearity on the left-hand side with the forcing on the right-hand side. It should be clear that a regular perturbation expansion of the form

\[ u = u_0 + \epsilon u_1 + \cdots \]

will fail at \( O(\epsilon) \) due to the forcing on the right-hand side since the initial conditions are arbitrary. This might lead one to conclude that

\[ u(t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n F_n(T, \tau), \quad \tau = \epsilon t, \quad T = t \left(1 + \sum_{n=2}^{\infty} \epsilon^n \omega_n\right). \]

As in class, we let

\[ g(t; \epsilon) = F_0(T, \tau) + \epsilon F_1(T, \tau) + o(\epsilon), \quad T = t \left[1 + O(\epsilon^2)\right]. \]

Substituting these expressions into (6.1), we have, to leading orders,

\[
\frac{\partial}{\partial T} \left[ \frac{\partial}{\partial T} (F_0 + \epsilon F_1) + \epsilon \frac{\partial F_0}{\partial \tau} \right] + \epsilon \frac{\partial}{\partial \tau} \left( \frac{\partial F_0}{\partial T} \right) + F_0 + \epsilon F_1 + \epsilon^2 F_2 - \epsilon (F_0 + \epsilon F_1)^2 = \epsilon \cos T.
\]

Expanding and taking only the terms to \( O(\epsilon) \), we have

\[ \frac{\partial^2 F_0}{\partial T^2} + F_0 = 0, \hspace{1cm} O(1) \]

\[ \frac{\partial^2 F_1}{\partial T^2} + 2 \frac{\partial^2 F_0}{\partial T \partial \tau} + F_1 - F_0^2 = \cos T, \hspace{1cm} O(\epsilon) \]
Solving our equations one at a time, we have
\[
F_0 = A_0(\tau) \cos T + B_0(\tau) \sin T,
\]
\[
\frac{\partial^2 F_1}{\partial T^2} + F_1 = (A_0 \cos T + B_0 \sin T)^2 + \cos T - 2(-A_0 \sin T + B_0' \cos T)
\]
\[
\frac{\partial^2 F_1}{\partial T^2} + F_1 = (1 - 2B_0') \cos T + \frac{A_0^2 + B_0'^2}{2} + A_0B_0 \sin 2T + \frac{(A_0^2 - B_0'^2) \cos 2T}{2} + 2A_0' \sin T.
\]
Therefore, we see that we have no way to suppress secularity, since to do so would make $B_0$ blow up as $\tau \to \infty$.

Hence we try to scale the problem. Since $B_0$ blew up in our first try, we expect large oscillations, so we let $u = \epsilon^{-\alpha} y$, where we expect $\alpha > 0$:
\[
\epsilon^{-\alpha} \ddot{y} + \epsilon^{-\alpha} y - \epsilon^{1-2\alpha} y^2 = \epsilon \cos t
\]
\[
\ddot{y} + y - \epsilon^{1-\alpha} y^2 = \epsilon^{1+\alpha} \cos t.
\]
We now closely examine the operator on the left-hand side. Since $\alpha < 0$, $\epsilon^{1+\alpha} \ll \epsilon^{1-\alpha} \ll 1$.

To simplify some of the algebra, we let
\[
\delta_1 = \epsilon^{1-\alpha}, \quad \delta_2 = \epsilon^{1+\alpha}
\]
into the above to obtain
\[
\ddot{y} + y - \delta_1 y^2 = \delta_2 \cos t.
\]
Since we are just trying to determine the appropriate scalings, for now we just assume the following initial conditions:
\[
y_0(0) = a, \quad \dot{y}(0) = 0.
\]
Letting $y(t; \delta_1) \sim y_0(t) + \delta_1 y_1(t) + \delta_1^2 y_2(t) + \cdots$, we have
\[
\frac{d^2}{dt^2} (y_0 + \delta_1 y_1 + \delta_1^2 y_2) + (y_0 + \delta_1 y_1 + \delta_1^2 y_2) + \delta_1 (y_0 + \delta_1 y_1)^2 = \delta_2 \cos t
\]
\[
\dot{y}_0 + y_0 = 0, \quad y_0(0) = a, \quad \dot{y}_0(0) = 0, \quad O(1)
\]
\[
\ddot{y}_1 + y_1 + y_0^2 = 0, \quad y_1(0) = 0, \quad \dot{y}_1(0) = 0. \quad O(\delta_1)
\]
Solving our equations in turn, we have
\[
y_0 = a \cos t,
\]
\[
\ddot{y}_1 + y_1 = -a^2 \cos^2 t = -\frac{a^2}{2} - \frac{a^2 \cos 2t}{2}
\]
\[
y_1 = -\frac{a^2}{2} + \frac{a^2 \cos 2t}{6} + A \cos t + B \sin t
\]
\[
= -\frac{a^2}{2} + \frac{a^2 \cos 2t}{6} + \frac{a^2 \cos t}{3}.
\]
Therefore, we see that the first secular-causing term from the operator won’t occur until \(O(\delta_1^2)\). This happens because we see that in (B) squaring a trigonometric term does not cause a problem. (The solution would still be okay at \(O(\epsilon)\) if we had both \(\sin\) and \(\cos\) terms.) We would like to suppress the secular-causing term from the right-hand side at the same time, so we let

\[
\delta_1^2 = \delta_2 \quad \implies \quad \epsilon^{2-2\alpha} = \epsilon^{1+\alpha} \quad \implies \quad \alpha = \frac{1}{3}.
\]

Substituting the above result into (A), we obtain

\[
\ddot{y} + y - \epsilon^{2/3} y^2 = \epsilon^{4/3} \cos t,
\]

and the correct expansion is given by

\[
u(t; \epsilon) = \epsilon^{-1/3} y(t; \epsilon), \quad y(t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{2n/3} F_n(T, \tau),
\]

\[
\tau = \epsilon^{4/3} t, \quad T = t \left(1 + \epsilon^{2/3} \omega_1 + \sum_{n=3}^{\infty} \epsilon^{2n/3} \omega_n\right),
\]

where the \(\epsilon^{4/3}\) scaling on \(\tau\) is motivated by the fact that the secularity doesn’t occur until the second order in the expansion.

Substituting this expression into (C), we have, to leading orders,

\[
(1 + \omega_1 \epsilon^{2/3}) \frac{\partial}{\partial T} \left[(1 + \omega_1 \epsilon^{2/3}) \frac{\partial}{\partial T} (F_0 + \epsilon^{2/3} F_1 + \epsilon^{4/3} F_2) + \epsilon^{4/3} \frac{\partial F_0}{\partial \tau}\right] + \epsilon^{4/3} \frac{\partial}{\partial \tau} \left(\frac{\partial F_0}{\partial T}\right)
\]

\[
+ F_0 + \epsilon^{2/3} F_1 + \epsilon^{4/3} F_2 - \epsilon^{2/3} (F_0 + \epsilon^{2/3} F_1)^2 = 0.
\]

Expanding and taking only the terms to \(O(\epsilon^{4/3})\), we have

\[
\frac{\partial^2 F_0}{\partial T^2} + F_0 = 0, \quad O(1)
\]

\[
\frac{\partial^2 F_1}{\partial T^2} + F_1 + 2 \omega_1 \frac{\partial^2 F_0}{\partial T \partial \tau} - F_0^2 = 0, \quad O(\epsilon^{2/3})
\]

\[
\frac{\partial^2 F_2}{\partial T^2} + F_2 + 2 \frac{\partial^2 F_0}{\partial T \partial \tau} + 2 \omega_1 \frac{\partial^2 F_1}{\partial T \partial T} - 2 F_0 F_1 + \omega_1 \frac{\partial^2 F_0}{\partial T^2} = \cos T, \quad O(\epsilon^{4/3})
\]

We need only the leading order of our boundary conditions:

\[
F_0(0, 0) = a, \quad \frac{\partial F_0}{\partial T}(0, 0) = 0.
\]

Now we solve our equations one at a time:

\[
F_0 = A_0(\tau) \cos T + B_0(\tau) \sin T, \quad A_0(0) = a, \quad B_0(0) = 0.
\]
\[
\frac{\partial^2 F_1}{\partial T^2} + F_1 = -2\omega_1 (A_0 \cos T + B_0 \sin T) + (A_0 \cos T + B_0 \sin T)^2
\]
\[
\frac{\partial^2 F_1}{\partial T^2} + F_1 = -2\omega_1 (A_0 \cos T + B_0 \sin T) + \frac{A_0^2 + B_0^2}{2} + A_0 B_0 \sin 2T + \frac{(A_0^2 - B_0^2) \cos 2T}{2}.
\]

We see that in order to suppress secularity we must have \(\omega_1 = 0\), and then we have
\[
F_1 = \frac{A_0^2 + B_0^2}{2} - \frac{A_0 B_0 \sin 2T}{3} - \frac{(A_0^2 - B_0^2) \cos 2T}{6} + A_1(\tau) \cos T + B_1(\tau) \sin T.
\]

Going to the next order and recalling that \(\omega_1 = 0\), we have
\[
\frac{\partial^2 F_2}{\partial T^2} + F_2 = \cos T - 2 \frac{\partial^2 F_0}{\partial T \partial \tau} + 2F_0 F_1
\]
\[
\frac{\partial^2 F_2}{\partial T^2} + F_2 = \cos T - 2 (B_0' \cos T - A_0' \sin T) + 2 (A_0 \cos T + B_0 \sin T) \times
\]
\[
\left[ \frac{A_0^2 + B_0^2}{2} - \frac{A_0 B_0 \sin 2T}{3} - \frac{(A_0^2 - B_0^2) \cos 2T}{6} \right] + \text{acceptable terms}
\]
\[
= 2 \left[ \frac{1}{2} - B_0' + \frac{A_0(A_0^2 + B_0^2)}{2} - \frac{A_0(A_0^2 - B_0^2)}{12} - \frac{A_0 B_0^2}{6} \right] \cos T
\]
\[
+ 2 \left[ A_0' + \frac{B_0(A_0^2 + B_0^2)}{2} + \frac{B_0(A_0^2 - B_0^2)}{12} - \frac{A_0^2 B_0}{6} \right] \sin T
\]
\[
+ \text{acceptable terms}.
\]

Therefore, we see that both bracketed terms must be set equal to zero to suppress secularity:
\[
B_0' = \frac{1}{2} + \frac{5A_0(A_0^2 + B_0^2)}{12}, \quad (D.1)
\]
\[
A_0' = -\frac{5B_0(A_0^2 + B_0^2)}{12}. \quad (D.2)
\]

Equations (D) cannot be easily solved in closed form; the phase plane is given below. The coordinates of the center are given where both \(A_0'\) and \(B_0'\) are zero, so we have
\[
-2^{1/2} = \frac{1}{2} + \frac{5A_0(A_0^2 + B_0^2)}{12},
\]
\[
0 = \frac{5B_0(A_0^2 + B_0^2)}{12}.
\]

Hence the solutions
\[
A_0(\tau) \equiv -\left( \frac{6}{5} \right)^{1/3}, \quad B_0(\tau) \equiv 0,
\]
are steady-states, as suggested in the remarks. The phase plane is shown below. Note that even when we start at the origin, we obtain a large oscillation.
2. Above is shown a graph of the \((\delta, \epsilon)\) plane for the standard Mathieu equation. Now consider the Mathieu equation with an extra term added:

\[ u'' + 2\beta u' + (\delta + \epsilon \cos t)u = 0, \quad |\epsilon| \ll 1, \]

(6.2)

where \(\beta\) is a constant. We wish to see how the stability diagram for \(\beta = 0\) changes when \(\beta \neq 0\).
(a) (3 points) By introducing a new variable \( v \), one can reduce the equation (6.2) for \( u \) to the standard Mathieu equation for \( v \) (perhaps with different parameters). Determine the relationship between \( u \) and \( v \).

**Solution.** The \( \beta \) term should introduce exponential growth or decay, so we let \( u = e^{\alpha t} v \) in (6.2) to obtain

\[
\alpha^2 e^{\alpha t} v + 2\alpha e^{\alpha t} v' + e^{\alpha t} v'' + 2\beta (\alpha e^{\alpha t} v + e^{\alpha t} v') + (\delta + \epsilon \cos t)e^{\alpha t} v = 0
\]

\[
v'' + 2(\beta + \alpha)v' + (\alpha^2 + 2\alpha\beta + \delta + \epsilon \cos t)v = 0.
\]

Therefore, by setting \( \alpha = -\beta \), we obtain

\[
v'' + (\delta - \beta^2 + \epsilon \cos t)v = 0.
\]

(b) (6 points) Explain why if \( \beta \) is “large,” the solution will be dominated by the linear damping/growth term. Determine “large” for each of the special values \( n^2/4 \) of \( \delta_0 \).

**Solution.** Since \( u = e^{-\beta t} v \), we see that the \( e^{-\beta t} \) term will dominate unless \( \beta t \) is on the order of the slow-time scales established for the standard Mathieu equation. Therefore, for \( n > 1 \), any \( \beta > O(\epsilon^2) \) will cause the linear term to dominate, and for \( n \leq 1 \), any \( \beta > O(\epsilon) \) will cause the linear term to dominate.