**Homework Set 4 Solutions**

1. (4 points) Show that in the model nonlinear example presented in class:

   \[ \epsilon y'' + y y' - y = 0, \quad y(0) = A, \quad y(1) = B, \]

   if we make the substitutions

   \[ B \mapsto -A, \quad A \mapsto -B, \quad x \mapsto 1 - x, \quad y \mapsto -y, \]

   we obtain the same differential equation as before.

   **Solution.** Let

   \[ \bar{x} = 1 - x, \quad \bar{y}(\bar{x}) = -y(x). \]

   Then our equations become

   \[ -\epsilon \frac{d^2 \bar{y}}{d\bar{x}^2} - \bar{y} \frac{d\bar{y}}{d\bar{x}} + \bar{y} = 0, \quad \bar{y}(1) = -A, \quad \bar{y}(0) = -B. \]

   Now letting

   \[ \bar{B} = -A, \quad \bar{A} = -B, \]

   we have

   \[ \epsilon \frac{d^2 \bar{y}}{d\bar{x}^2} + \bar{y} \frac{d\bar{y}}{d\bar{x}} - \bar{y} = 0, \quad \bar{y}(0) = \bar{A}, \quad \bar{y}(1) = \bar{B}, \]

   as required.

2. Consider the following problem:

   \[ x^3 y' = (\epsilon x + \epsilon^2 x + 2\epsilon^3)y^2, \quad 0 \leq x \leq 1, \quad 0 < \epsilon \ll 1, \quad (4.1) \]

   \[ y(1) = 1 - \epsilon. \]

   (a) (2 points) Using the equation, find \( y(0) \).

   **Solution.** It is clear that for small \( x \), the dominant balance is

   \[ x^3 y' = 2\epsilon^3 y^2 \]

   for any value of \( \epsilon \). Letting \( y = x^n \) in the above, we see that for a balance \( n = 2 \), and hence \( y(0) = 0 \).

   (b) (5 points) Construct any needed outer expansions to \( O(\epsilon^2) \).
Solution. Letting \( y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + o(\epsilon) \), we have
\[
x^3(y'_0 + \epsilon y'_1 + \epsilon^2 y'_2) = (\epsilon x + \epsilon^2 x + 2\epsilon^3)(y_0 + \epsilon y_1)^2
\]
\[
x^3y'_0 = 0, \quad y_0(0) = 0, \quad y_0(1) = 1, \quad O(1)
\]
\[
x^3y'_1 = xy'_0, \quad y_1(0) = 0, \quad y_1(1) = -1, \quad O(\epsilon)
\]
\[
x^3y'_2 = xy'_0 + 2xy_0y_1, \quad y_1(0) = 0, \quad y_1(1) = 0, \quad O(\epsilon^2)
\]
We see that we have two possible candidates. If we choose to solve the boundary condition at 0, we have \( y_0 = y_1 = y_2 \equiv 0 \). If we choose to solve the boundary condition at 1, we have
\[
y'_0 = 0 \quad \implies \quad y_0(x) = 1,
\]
\[
y'_1 = \frac{1}{x^2}, \quad \implies \quad y_1(x) = -\frac{1}{x},
\]
\[
y'_2 = \frac{1}{x^2} - \frac{2}{x^3}, \quad \implies \quad y_2(x) = -\frac{1}{x} + \frac{1}{x^2}.
\]
Therefore, we have two choices:
\[
y_L(x) = 0, \quad y_R(x) = 1 - \frac{\epsilon}{x} + \frac{\epsilon^2(1-x)}{x^2}.
\]
(c) (13 points) Construct any needed inner expansions to \( O(\epsilon) \).

Solution. We begin by assuming that \( y_L \) is correct; that is, that our boundary layer is near \( x = 1 \). Letting
\[
\bar{x} = \frac{x - 1}{\epsilon^n}, \quad y(x) \sim \bar{y}(\bar{x}),
\]
we have, to leading order,
\[
\epsilon^{-n} \frac{d\bar{y}}{d\bar{x}} = \epsilon \bar{y}'^2,
\]
for which there is no dominant balance. Therefore, we see that we must scale near \( x = 0 \):
\[
\xi = \frac{x}{\epsilon^m}, \quad y(x) \sim w(\xi).
\]
Using these scalings, we have
\[
\epsilon^{2m} \xi^3 \frac{dw}{d\xi} = (\epsilon^{1+m} \xi + \epsilon^{2+m} \xi + 2\epsilon^3)w^2,
\]
which has two dominant balances: \( m = 1 \) and \( m = 2 \).

At this order, the only meaningful balance is \( m = 1 \), so we have, letting \( w(\xi) = w_0(\xi) + \epsilon w_1(\xi) \),
\[
\xi^3 \left( \frac{dw_0}{d\xi} + \epsilon \frac{dw_1}{d\xi} \right) = (\xi + \epsilon \xi + 2\epsilon)(w_0^2 + 2\epsilon w_0 w_1),
\]
\[
\xi^3 \frac{dw_0}{d\xi} = \xi w_0^2, \quad w_0(0) = 0, \quad w_0(\infty) = y_0(0) = 1, \quad O(1)
\]
\[
\xi^3 \frac{dw_1}{d\xi} = (\xi + 2)w_0^2 + 2\xi w_0 w_1, \quad w_1(0) = 0. \quad O(\epsilon)
\]

Solving the leading order equation, we have

\[
-\frac{1}{w_0} = -\frac{1}{\xi} - A \quad \implies \quad w_0(\xi) = \frac{\xi}{1 + A\xi} \quad \implies \quad w_0(\xi) = \frac{\xi}{1 + \xi}.
\]

Then the next order equation becomes

\[
\frac{\xi^3}{\xi^2} \frac{dw_1}{d\xi} + \left[ \frac{2(1 + \xi)}{\xi^2} - \frac{(1 + \xi)^2}{\xi^3} \right] w_1 = \frac{(\xi + 2)}{\xi^3} \frac{(1 + \xi)^2}{\xi^2} w_1 = -\frac{1}{\xi} - \frac{1}{\xi^2} + A
\]

\[
\Rightarrow w_1(\xi) = -\frac{1}{1 + \xi} - \frac{A\xi^2}{(1 + \xi)^2}.
\]

The matching is tricky, so we introduce the intermediate variable \(x_\eta\). Rewriting \(w\) and \(y\) in these variables, we have

\[
w(x_\eta) \sim \frac{x_\eta \eta / \epsilon}{1 + x_\eta \eta / \epsilon} - \frac{\epsilon}{1 + x_\eta \eta / \epsilon} + \frac{A(x_\eta \eta)^2 / \epsilon^2}{(1 + x_\eta \eta / \epsilon)^2}
\]

\[
\sim \frac{1}{1 + \epsilon / x_\eta \eta} - \frac{x_\eta \eta (1 + \epsilon / x_\eta \eta)}{1 + x_\eta \eta / \epsilon} + \epsilon \left(1 + \epsilon / x_\eta \eta\right)^2
\]

\[
\sim 1 - \frac{\epsilon}{x_\eta \eta} + \frac{\epsilon^2}{x_\eta \eta^2} - \frac{\epsilon^2}{x_\eta \eta} + \epsilon A \left(1 - \frac{2 \epsilon}{x_\eta \eta}\right)
\]

\[
y(x_\eta) \sim 1 - \frac{\epsilon}{x_\eta \eta} - \epsilon^2 \left[ \frac{1}{x_\eta \eta} - \frac{1}{(x_\eta \eta)^2}\right].
\]

To match, we see that we must have \(A = 0\).

However, we note that \(w_1(0) \neq 0\), so we must have an additional boundary layer near \(x = 0\) with our other balance, namely \(m = 2\). The scaling is at \(O(\epsilon)\) in our dependent variable, so we let

\[
\zeta = \frac{x}{\epsilon^2}, \quad y(x) \sim \epsilon f(\zeta),
\]
and expand only to leading order, which yields

\[ \zeta^3 \frac{df}{d\zeta} = (\zeta + 2)f^2, \quad f(0) = 0, \quad f(\infty) = w_1(0) = -1 \quad O(\epsilon^5) \]

\[ \frac{-1}{f} = \frac{1}{\zeta} - \frac{1}{\zeta^2} - B \]

\[ f = \frac{\zeta^2}{B\zeta^2 + \zeta + 1} \]

\[ f(\zeta) = \frac{\zeta^2}{-\zeta^2 + \zeta + 1}. \]

(d) (3 points) Construct the uniform expansion to \( O(\epsilon) \).

Solution. We begin by writing all our answers in the same variables:

\[ y_R(x) = 1 - \frac{\epsilon}{x} + \frac{\epsilon^2(1 - x)}{x^2} \]

\[ w(x/\epsilon) = \frac{x/\epsilon}{1 + x/\epsilon} - \frac{\epsilon}{1 + x/\epsilon} = \frac{x - \epsilon^2}{x + \epsilon} \]

\[ f(x/\epsilon^2) = \epsilon \frac{x^2/\epsilon^4 + x/\epsilon^2 + 1}{-x^2/\epsilon^4 + x/\epsilon^2 + 1} = \frac{\epsilon x^2}{\epsilon^4 + x\epsilon^2 - x^2}. \]

We see from our previous matching that we have matched the entire outer solution, and so the outer solution is the common part, as is \(-\epsilon\) from the matching of the inner and intermediate layers. Therefore, we have

\[ y_u = \frac{x - \epsilon^2}{x + \epsilon} + \frac{\epsilon x^2}{\epsilon^4 + x\epsilon^2 - x^2} + \epsilon. \]

3. Consider the following singular boundary-value problem:

\[ \epsilon y'' + \frac{y'}{x} - 2y = 0, \quad \epsilon^2 \leq x \leq 1, \quad 0 < \epsilon \ll 1, \quad (4.2a) \]

\[ y(\epsilon^2) = 1, \quad y(1) = 1. \quad (4.2b) \]

(a) (3 points) Construct any needed leading-order outer expansions.

Solution. Letting \( y \sim y_0 + o(1) \) and letting \( \epsilon \to 0 \), we have

\[ \frac{y_0'}{x} - 2y_0 = 0, \quad y(0) = 1, \quad y(1) = 1, \quad y_0 = Ae^{x^2}, \quad y_0 = e^{x^2 - 1}, \]

where we have satisfied the right-hand boundary because it isn’t the singular one.
(b) (5 points) Construct any needed leading-order inner expansions.

*Solution.* Inserting the obvious scalings, we obtain

\[ \xi = \frac{x}{\epsilon^2}, \quad y(x) \sim w(\xi), \]

\[
\epsilon^{-3} w'' + \epsilon^{-4} \frac{w'}{\xi} - 2w = 0 \\
\epsilon w'' + \frac{w'}{\xi} - 2\epsilon^4 w = 0, \quad w(1) = 1, \quad w(\infty) = e^{-1}. \tag{A}
\]

The leading order solution is a constant which can’t satisfy the boundary conditions. Therefore, we introduce yet another boundary layer:

\[ \zeta = \frac{\xi - 1}{\epsilon^a}, \quad w(\xi) \sim f(\zeta). \]

Using these scalings, equations (A) become, to leading order,

\[
\epsilon^{1-2a} f'' + \epsilon^{-a} \frac{f'}{1 + \epsilon^a \zeta} - 2\epsilon^4 f = 0 \quad \implies \quad a = 1 \\
f'' + f' = 0, \quad f(1) = 1, \quad f(\infty) = e^{-1}, \\
f(\zeta) = e^{-1} + (1 - e^{-1})e^{-\zeta}. \]

(c) (5 points) Construct and sketch the leading-order uniformly valid approxima-
tion of the solution(s).

*Solution.* Since \( a = 1 \), we have that

\[ \zeta = \frac{\xi - 1}{\epsilon} = \frac{x - \epsilon^2}{\epsilon^3}. \]

Writing the inner solution in terms of \( x \), adding it to the outer solution, and subtracting
the common part, which is \( e^{-1} \), we have

\[ y_u = e^{x^2 - 1} + (1 - e^{-1}) \exp \left( \frac{\epsilon^2 - x}{\epsilon^3} \right). \]

Here’s a graph with \( \epsilon = 0.01 \).