Homework Set 5

Read sections 9.5, 11.1–11.3.

Higher-Order ODEs

1. Consider the following system:

\[ \epsilon v'' + u'v' = 0, \quad u(0) = u(1) = 0, \quad (5.1a) \]
\[ u'' = v, \quad v(0) = 2v_\star > 0, \quad v(1) = 2\alpha^2v_\star, \quad \alpha > 1. \quad (5.1b) \]

(a) (2 points) Explain why we shouldn’t expect any layers in \( u \) or \( u' \).
(b) (5 points) Construct two sets of outer solutions \((u, v)\) that satisfy three of the four boundary conditions in (5.1). Show that it is impossible to construct a bounded inner solution for either set.

Hence we must have an interior layer.

(c) (5 points) Show that the interior layer must be about \( x = \alpha/(\alpha + 1) \), and write down the corresponding outer solutions for \( u \) and \( v \).

(d) (3 points) Sketch the solution.

2. Now we wish to examine the system (5.1a) coupled to

\[ u'' = -v, \quad v(0) = 2(\bar{v} - v_\star), \quad v(1) = 2(\bar{v} + v_\star), \quad \bar{v} > v_\star > 0, \quad (5.2) \]

which replaces (5.1b). You may assume without proof that this system has only boundary layers.

(a) (3 points) Using the equations only, show that \( v'(0) = v'(1) \) for any \( \epsilon \).
(b) (4 points) Use part (a) to determine the proper scalings and identify the position for each needed boundary layer. Explain why your choice of layers is unique.

(c) (5 points) Write down a leading-order uniformly valid solution for \( v \) and the leading-order outer solution for \( u \).

(d) (3 points) Sketch the solution.
Linear 2-Timing

3. (10 points) When performing two-timing, sometimes it is more convenient to write the fast-time solution as

\[ F(T, \tau) = A(\tau) \cos(T + \phi(\tau)) \]  \hspace{1cm} (5.3)

instead of

\[ F(T, \tau) = A(\tau) \cos T + B(\tau) \sin T. \]

For the example given in class:

\[ \ddot{x} + 2\epsilon \dot{x} + x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1, \]

repeat the analysis using (5.3) and verify that you obtain the same result as that derived in class.