

Homework Set 9 Solutions

1. (4 points) Find the leading-order behavior of

$$I(x) = \int_{-2}^2 t e^{x \sin t} dt, \quad x \rightarrow \infty.$$

Solution. Using Laplace's method, we have that

$$I(x) = \int_{-2}^2 f(t) e^{x\phi(t)} dt, \quad f(t) = t, \quad \phi(t) = \sin t, \quad \phi'(t) = \cos t.$$

We see that $\phi'(t_*) = 0$ when $t_* = \pm\pi/2$. At these points, we have

$$\begin{aligned} \phi''(\pi/2) = -\sin \pi/2 = -1 &\implies \phi(\pi/2) = 1 \text{ is a maximum,} \\ \phi''(-\pi/2) = -\sin -\pi/2 = 1 &\implies \phi(-\pi/2) = -1 \text{ is a minimum.} \end{aligned}$$

Then using the formula from class, we have

$$I(x) \sim \sqrt{\frac{2\pi}{x|\phi''(\pi/2)|}} f(\pi/2) \exp(x\phi(\pi/2)) = e^x \sqrt{\frac{2\pi}{x}} \left(\frac{\pi}{2}\right) = \frac{e^x \pi^{3/2}}{\sqrt{2x}}.$$

2. (6 points) Find the first term in the asymptotic expansion of

$$I(x) = \int_0^\infty t^x e^{-t} \log t dt, \quad x \rightarrow \infty.$$

Solution. Analyzing the integral as it now stands, we see that $t^x = e^{x \log t}$, and hence the dominant contribution to this part of the integrand comes from $t = \infty$, where the rest of the integrand is transcendentally small. Combining the exponentials together, we have

$$t^x e^{-t} = e^{x \log t - t},$$

which can't be written in the standard Laplace's method form. But we note that if we write

$$t^x e^{-t} = e^{x\phi(t,x)}, \quad \phi(t,x) = x \log t - t, \quad \frac{\partial \phi}{\partial t} = \frac{x}{t} - 1,$$

then we see that the partial derivative is zero whenever $t = x$. This has all the hallmarks of a moving maximum problem, so we let $t = sx$ to obtain

$$I(x) = \int_0^\infty (sx)^x e^{-sx} \log(sx)(x ds) = x^{x+1} \int_0^\infty e^{x(\log s - s)} (\log x + \log s) ds. \quad (\text{A})$$

Therefore, we consider the following integral:

$$J = \int_0^\infty f(s)e^{x\phi(s)} ds, \quad \phi(s) = \log s - s, \quad \phi'(s) = \frac{1}{s} - 1, \quad \phi''(s) = -\frac{1}{s^2}.$$

$$\phi'(1) = 0, \quad \phi(1) = \phi''(1) = -1.$$

Therefore, using Laplace's method, we have that

$$J \sim \sqrt{\frac{2\pi}{x}} f(1)e^{-x}, \quad x \rightarrow \infty.$$

Since $\log 1 = 0$, we see that the second term would require further terms in Watson's Lemma and would be subdominant. Hence only the first term in the parentheses in (A) contributes and we have

$$I(x) \sim x^{x+1} \log x \left(\sqrt{\frac{2\pi}{x}} e^{-x} \right) = x^{x+1/2} e^{-x} \log x \sqrt{2\pi}, \quad x \rightarrow \infty.$$

3. Consider the following function:

$$H(\beta) = \frac{2\beta}{\pi} \int_0^\infty t e^{-t^{3/2}} \sin \beta t dt, \quad \beta \in \mathcal{R}^+.$$

We begin by considering the case where $\beta \rightarrow 0^+$.

(a) (4 points) Introduce a substitution that transforms $H(\beta)$ into standard Watson's Lemma form.

Solution. We wish the exponential to be of the form $e^{-\lambda u}$, where $\lambda \rightarrow \infty$. In addition, we wish the argument of the sine to be a function of u only. Therefore, we let

$$u = (\beta t)^{3/2}, \quad t = \frac{u^{2/3}}{\beta}, \quad dt = \frac{2 du}{3\beta u^{1/3}}$$

in order to obtain

$$\begin{aligned} H(\beta) &= \frac{2\beta}{\pi} \int_0^\infty \frac{u^{2/3}}{\beta} e^{-u/\beta^{3/2}} \sin \beta \left(\frac{u^{2/3}}{\beta} \right) \frac{2 du}{3\beta u^{1/3}} \\ &= \frac{4}{3\pi\beta} \int_0^\infty u^{1/3} e^{-u/\beta^{3/2}} \sin u^{2/3} du. \end{aligned}$$

(b) (3 points) Find the first term in the asymptotic expansion of $H(\beta)$ as $\beta \rightarrow 0$.

Solution. Using Watson's Lemma considering β^{-1} to be large, we have

$$H(\beta) \sim \frac{4}{3\pi\beta} \int_0^\infty u e^{-u/\beta^{3/2}} = \frac{4}{3\pi\beta} \frac{\Gamma(2)}{(\beta^{-3/2})^2} = \frac{4\beta^2}{3\pi}.$$

Next we consider the case where $\beta \rightarrow \infty$.

- (c) (4 points) Introduce a substitution that transforms $H(\beta)$ into standard stationary phase form with nonmonotonic exponent.

Solution. We rewrite $H(\beta)$ slightly:

$$H(\beta) = \frac{2\beta}{\pi} \Im \int_0^\infty t e^{-t^{3/2} + i\beta t} dt.$$

In order to attain stationary phase form, we must have an i in the argument of the exponential, as well as the same power of β in both portions. Thus, letting $t = -\beta^\alpha u$, we see that $\alpha = 2$ and we have

$$H(\beta) = \frac{2\beta}{\pi} \Im \int_0^{-\infty} (-\beta^2 u) e^{-\beta^3 (-u)^{3/2} - i\beta^3 u} (-\beta^2 du) = -\frac{2\beta^5}{\pi} \Im \int_{-\infty}^0 u e^{i\beta^3 (u^{3/2} - u)} du.$$

- (d) (7 points) Find the first term in the asymptotic expansion of $H(\beta)$ as $\beta \rightarrow \infty$.

Solution. In our case we have

$$H(\beta) = -\frac{2\beta^5}{\pi} \Im \int_{-\infty}^0 u e^{i\beta^3 \phi(u)} du, \quad \phi(u) = u^{3/2} - u, \quad \phi'(u) = \frac{3u^{1/2}}{2} - 1.$$

The stationary point is $u = 4/9$, which is outside the range of integration. Therefore, the next thing to try is integration by parts. Unfortunately, due the lack of analyticity in the exponent, integration by parts also fails. To make the exponent analytic, we let $u = v^2$ to obtain

$$H(\beta) = -\frac{2\beta^5}{\pi} \Im \int_{-i\infty}^0 v^2 e^{i\beta^3 (v^3 - v^2)} (2v dv) = -\frac{4\beta^5}{\pi} \Im \int_{-i\infty}^0 v^3 e^{i\beta^3 (v^3 - v^2)} dv,$$

where the negative square root has been chosen so the integral remains convergent. But now we have

$$\phi'(v) = 3v^2 - 2v \quad \implies \quad v = 0, 2/3.$$

Thus the stationary point $v = 0$ is an endpoint, and we have

$$\phi''(0) = -2.$$

Therefore, expanding in this neighborhood, we have

$$H(\beta) \sim -\frac{4\beta^5}{\pi} \Im \int_{-i\infty}^0 v^3 (1 + i\beta^3 v^3) e^{-i\beta^3 v^2} dv.$$

We may rotate the contour to the ray $\arg(v) = -\pi/4$ since there are no singularities. Then we have

$$H(\beta) \sim -\frac{4\beta^5}{\pi} \Im \int_{e^{-\pi i/4}\infty}^0 v^3 (1 + i\beta^3 v^3) e^{-i\beta^3 v^2} dv.$$

Letting $v = e^{-\pi i/4}r$, we have

$$\begin{aligned}
 H(\beta) &\sim -\frac{4\beta^5}{\pi} \Im \int_{-\infty}^0 e^{-3\pi i/4} r^3 (1 + ie^{-3\pi i/4} \beta^3 r^3) e^{-i\beta^3(-ir^2)} (e^{-\pi i/4} dr) \\
 &= -\frac{4\beta^5}{\pi} \Im \int_0^{\infty} r^3 (1 + ie^{-3\pi i/4} \beta^3 r^3) e^{-\beta^3 r^2} dr = -\frac{4\beta^8}{\pi} \Re \int_0^{\infty} e^{-3\pi i/4} r^6 e^{-\beta^3 r^2} dr \\
 &= \frac{2\beta^8 \sqrt{2}}{\pi} \int_0^{\infty} r^6 e^{-\beta^3 r^2} dr = \frac{2\beta^8 \sqrt{2}}{\pi} \int_0^{\infty} x^3 e^{-\beta^3 x} \frac{dx}{2\sqrt{x}} \\
 &= \frac{\beta^8 \sqrt{2}}{\pi} \frac{\Gamma(7/2)}{(\beta^3)^{7/2}} = \frac{\sqrt{2}}{\pi \beta^{5/2}} \left[\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma(1/2) \right] = \frac{15}{4\beta^{5/2} \sqrt{2\pi}}.
 \end{aligned}$$

4. (12 points) Consider the following integral:

$$I = \int_a^b f(t) e^{ik\phi(t)} dt.$$

Here f and ϕ are well-defined throughout the range. The only critical point for ϕ occurs at a point α , where α is not an endpoint. At α , $\phi''(\alpha) = 0$, $\phi^{(3)}(\alpha) > 0$, $f(\alpha) \neq 0$. Prove that

$$I \sim \frac{\Gamma(1/3) f(\alpha) e^{ik\phi(\alpha)}}{\sqrt{3}} \left[\frac{6}{k\phi^{(3)}(\alpha)} \right]^{1/3} + O(k^{-2/3}).$$

Solution. We first consider the range of integration away from the critical point:

$$I = \int_{\alpha-\epsilon}^{\alpha+\epsilon} f(t) e^{ik\phi(t)} dt + \int_{|t-\alpha|>\epsilon} f(t) e^{ik\phi(t)} dt.$$

But from notes in class we have that the second integral is $O(k^{-1})$. Letting $y = t - \alpha$ and expanding for small y , we have

$$I \sim \int_{-\epsilon}^{\epsilon} f(\alpha + y) \exp \left\{ ik \left[\phi(\alpha) + \frac{y^3 \phi^{(3)}(\alpha)}{3!} + \frac{y^4 \phi^{(4)}(\alpha)}{4!} \right] \right\} dy.$$

Note that we have kept one more term in the expansion of the exponential so that we may check what the error is in truncating it. However, we still only wish to keep the leading-nonconstant term in the exponential when performing stationary phase, so we choose $\epsilon^4 k \ll 1$ so we have

$$\begin{aligned}
 I &\sim e^{ik\phi(\alpha)} \int_0^{\epsilon} [f(\alpha) - f'(\alpha)y_+] \exp \left[-ik \frac{y_+^3 \phi^{(3)}(\alpha)}{3!} \right] \left[1 + \frac{ky_+^4 \phi^{(4)}(\alpha)}{4!} \right] dy_+ \\
 &\quad + e^{ik\phi(\alpha)} \int_0^{\epsilon} [f(\alpha) + f'(\alpha)y_-] \exp \left[ik \frac{y_-^3 \phi^{(3)}(\alpha)}{3!} \right] \left[1 + \frac{ky_-^4 \phi^{(4)}(\alpha)}{4!} \right] dy_-,
 \end{aligned}$$

where we have expanded $f(\alpha + y)$ and we have broken up the region into pieces.

We now wish to transform the integral into Watson's Lemma form. To do so, we must introduce two separate transformations, denoted by the \pm subscript, for the two integrals. Summarizing, we have

$$z_{\pm} = \pm ik \frac{y_{\pm}^3 \phi^{(3)}(\alpha)}{3!}, \quad dy_{\pm} = \frac{e^{\mp \pi i/6} dz_{\pm}}{3} \left[\frac{6}{\phi^{(3)}(\alpha) z_{\pm}^2} \right]^{1/3}.$$

$$\begin{aligned} I \sim & \frac{f(\alpha) e^{-\pi i/6}}{3} \left[\frac{6}{\phi^{(3)}(\alpha)} \right]^{1/3} e^{ik\phi(\alpha)} \int_0^{i\Delta} [1 + O(z_+^{1/3}) + O(kz_+^{4/3})] \frac{e^{-kz_+}}{z_+^{2/3}} dz_+ \\ & + \frac{f(\alpha) e^{\pi i/6}}{3} \left[\frac{6}{\phi^{(3)}(\alpha)} \right]^{1/3} e^{ik\phi(\alpha)} \int_0^{-i\Delta} [1 + O(z_-^{1/3}) + O(kz_-^{4/3})] \frac{e^{-kz_-}}{z_-^{2/3}} dz_-, \end{aligned}$$

where Δ is a positive real constant. We next rotate each of the contours and let $\Delta \rightarrow \infty$. But then the integrals are the same, so we have

$$I \sim \frac{f(\alpha) e^{ik\phi(\alpha)}}{3} \left[\frac{6}{\phi^{(3)}(\alpha)} \right]^{1/3} \left(e^{-\pi i/6} + e^{\pi i/6} \right) \int_0^{\infty} [z_+^{-2/3} + O(z_+^{-1/3}) + O(kz_+^{2/3})] e^{-kz_+} dz_+,$$

where we again we have made an $O(k^{-1})$ error by taking $\Delta \rightarrow \infty$.

Now using Watson's Lemma, we have

$$\begin{aligned} I \sim & \frac{f(\alpha)}{3} \left[\frac{6}{\phi^{(3)}(\alpha)} \right]^{1/3} e^{ik\phi(\alpha)} (2 \cos \pi/6) \left[\frac{\Gamma(1/3)}{k^{1/3}} + O(k^{-2/3}) + O(k^{-2/3}) \right] \\ \sim & \frac{f(\alpha) \Gamma(1/3)}{\sqrt{3}} \left[\frac{6}{k\phi^{(3)}(\alpha)} \right]^{1/3} e^{ik\phi(\alpha)} + O(k^{-2/3}), \end{aligned}$$

as required.