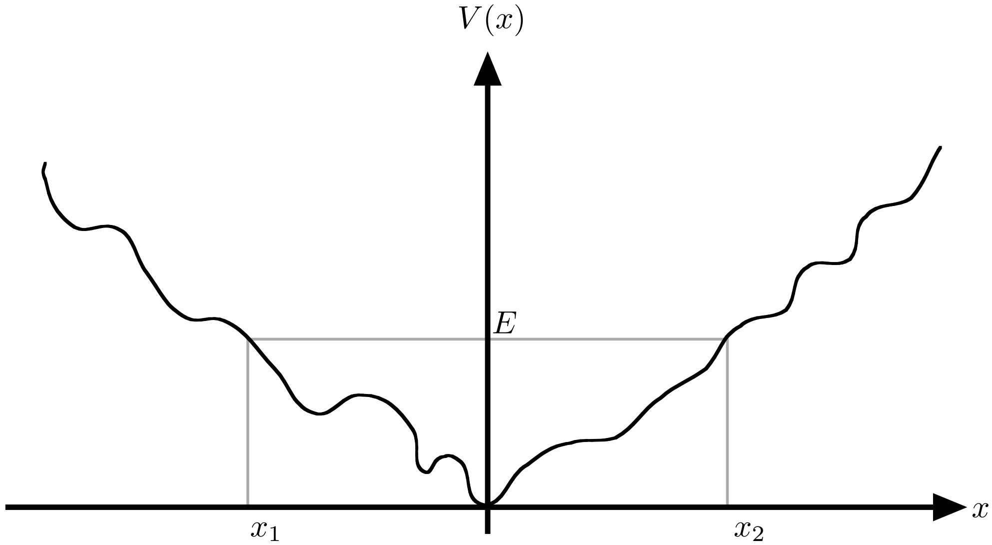
MATH 810-010 Asymptotics and Perturbation Methods
Prof. D. A. Edwards Due: Apr. 19, 2010

Homework Set 8 Solutions



1. Consider the following problem:

$$u'' + \lambda^2[E - V(x)]u = 0, \qquad -\infty < x < \infty, \quad \lambda \to \infty,$$

where $V(x)$ is as shown above. We wish to analyze bounded solutions of this operator.

(a) (4 points) Find the outer solutions to the problem. (*Hint: You need not repeat all the analysis; just make sure have all the variables defined in an appropriate way.*)

Solution. The problem is written in classical WKB form. Therefore, we see that our outer exponential solutions are as follows:

$$u_{\mathrm{L}} = \frac{ce^z}{[V(x) - E]^{1/4}}, \qquad z = \lambda \int_{x_1}^{x} \sqrt{|V(t) - E|}\,dt, \quad x < x_1$$

$$u_{\mathrm{R}} = \frac{d}{[V(x) - E]^{1/4}} \exp\left(-\lambda \int_{x_2}^{x} \sqrt{V(t) - E}\right) dt, \quad x > x_2,$$

where in the second line we have used the fact that $V(t) > E$ in that region. Note the difference in the lower limit of integration in our variables. This is done to ensure that our solutions decay as $x \to \pm\infty$. For the oscillatory region, we have

$$u_{\mathrm{I}} = \frac{1}{[E - V(x)]^{1/4}} \left[a \cos\left(z + \frac{\pi}{4}\right) + b \sin\left(z + \frac{\pi}{4}\right)\right], \quad x_1 < x < x_2,$$

where the x_1 lower limit has been chosen for notational convenience and we use the fact that in the interior, $E < V(x)$.

(b) (5 points) Using the matching formulas given in class, construct the full solution to the problem.

Solution. Near x_1, the problem goes from exponential to oscillatory, so we use the matching formulas (5+). Hence we have that

$$u_{\mathrm{I}} = \frac{2c}{[E - V(x)]^{1/4}} \cos\left(-z + \frac{\pi}{4}\right),$$

where we have to use the minus sign in (5+). But $\cos(-z + \pi/4) = \sin(z + \pi/4)$, so we have

$$u_{\mathrm{I}} = \frac{2c}{[E - V(x)]^{1/4}} \sin\left(z + \frac{\pi}{4}\right).$$

Near x_2, it is convenient to write our solution u_{I} in the following way:

$$u_{\mathrm{I}} = \frac{2c}{[E - V(x)]^{1/4}} \sin\left(\lambda \int_{x_2}^{x} \sqrt{E - V(t)}\, dt + \alpha + \frac{\pi}{4}\right), \text{ where} \tag{A}$$

$$\alpha = \lambda \int_{x_1}^{x_2} \sqrt{E - V(t)}\, dt.$$

Performing the matching at x_2, we see that to match from oscillatory to exponential we must have

$$u_{\mathrm{I}} = \frac{2d}{[E - V(x)]^{1/4}} \cos\left(\lambda \int_{x_2}^{x} \sqrt{E - V(t)}\, dt + \frac{\pi}{4}\right). \tag{B}$$

But our two equations (A) and (B) only make sense and match when

$$d = \pm c, \qquad \sin(z + \alpha + \pi/4) = \pm\cos(z + \pi/4) = \pm\sin(-z + \pi/4). \tag{C}$$

(c) (2 points) Show that a solution can exist for $x \in [x_1, x_2]$ only for values E_n of E given by

$$\int_{x_1}^{x_2} [E_n - V(x)]^{1/2}\, dx = \frac{(n + 1/2)\pi}{\lambda}.$$

Solution. From (C), we have that

$$\begin{aligned}
\sin(z + \alpha + \pi/4) &= \mp\sin(z - \pi/4) \\
z + \alpha + \frac{\pi}{4} &= z - \frac{\pi}{4} + n\pi, \qquad n \in \mathcal{Z} \\
\alpha &= \left(n + \frac{1}{2}\right)\pi \\
\int_{x_1}^{x_2} \sqrt{E - V(t)}\, dt &= \frac{(n + 1/2)\pi}{\lambda},
\end{aligned}$$

as required.

2. Consider the following problem:

$$w'' - k^2 x(x^2+1)^2 w = 0, \quad x \geq -1, \qquad w(-1) = 0, \qquad w(\infty) = 0, \qquad k > 0.$$

(a) (5 points) Find the outer solutions to the problem.

Solution. The problem is written in classical WKB form. Here

$$z = \int_0^x k(t^2+1)\sqrt{|t|}\,dt = \operatorname{sgn}(x) k \left(\frac{2|x|^{7/2}}{7} + \frac{2|x|^{3/2}}{3}\right).$$

Therefore, we see that our outer exponential solution is

$$w_{\mathrm{R}} = \frac{de^{-z}}{|x|^{1/4}\sqrt{x^2+1}}, \quad x > 0,$$

while the trigonemtric solution is

$$w_{\mathrm{L}} = \frac{1}{|x|^{1/4}\sqrt{x^2+1}}\left[a\cos\left(z+\frac{\pi}{4}\right) + b\sin\left(z+\frac{\pi}{4}\right)\right], \quad -1 \leq x \leq 0.$$

where we have left the constants arbitrary for now.

(b) (5 points) Using the matching formulas given in class, construct the full solution to the problem. Be sure to express your final solution in terms of x.

Solution. Here $Q < 0$, so we use the matching formulas (5−). Since $c = 0$, we see that our matching formulas become

$$b = 0, \qquad a = 2d.$$

Then incorporating the $\operatorname{sgn}(x)$ term in z, we have

$$\begin{aligned}
w_{\mathrm{R}} &= \frac{d}{|x|^{1/4}\sqrt{x^2+1}}\exp\left(-k\left(\frac{2|x|^{7/2}}{7} + \frac{2|x|^{3/2}}{3}\right)\right), \\
w_{\mathrm{L}} &= \frac{2d}{|x|^{1/4}\sqrt{x^2+1}}\cos\left(-k\left(\frac{2|x|^{7/2}}{7} + \frac{2|x|^{3/2}}{3}\right) + \frac{\pi}{4}\right) \\
&= \frac{2d}{|x|^{1/4}\sqrt{x^2+1}}\cos\left(k\left(\frac{2|x|^{7/2}}{7} + \frac{2|x|^{3/2}}{3}\right) - \frac{\pi}{4}\right),
\end{aligned}$$

where we have used the symmetry of the cosine.

(c) (3 points) Show that a nontrivial solution can exist only for values of k approximately equal to

$$\frac{21(4n-1)\pi}{80}, \qquad n \in \mathcal{Z}^+.$$

Solution. For $w_{\mathrm{L}}(-1) = 0$ and $d \neq 0$, we must have that

$$\begin{aligned}
\cos\left(k\left(\frac{2}{7}+\frac{2}{3}\right)-\frac{\pi}{4}\right) &= 0\\
\frac{20k}{21}-\frac{\pi}{4} &= \left(n-\frac{1}{2}\right)\pi, \qquad n \in \mathcal{Z}\\
\frac{80k}{21} &= (4n-1)\pi\\
k &= \frac{21(4n-1)\pi}{80}, \qquad n \in \mathcal{Z}^+
\end{aligned}$$

where we impose the additional requirement since $k > 0$.

3. Consider the function

$$f(z) = \int_0^\infty \frac{e^{-zu}}{1+u^4}\,du.$$

(a) (2 points) Find the region of analyticity for $f(z)$.

Solution. It is clear that $f(z)$ is a convergent integral whenever $\Re(z) \geq 0$.

(b) (4 points) Find its complete asymptotic expansion as $z \to \infty$ in this region.

Solution. Using Watson's Lemma, we have

$$f(z) = \int_0^\infty e^{-zu}\sum_{n=0}^\infty (-u^4)^n\,du = \sum_{n=0}^\infty \frac{(-1)^n\Gamma(4n+1)}{z^{4n+1}}.$$

(c) (7 points) Show that

$$g(z) = \frac{\pi(1+i)}{2\sqrt{2}}\exp\left(ze^{3\pi i/4}\right) - i\int_0^\infty \frac{e^{izv}}{1+v^4}\,dv$$

is the analytic continuation of $f(z)$ to the region $0 < \arg z < \pi$.

Solution. Letting $v = iu$, we obtain

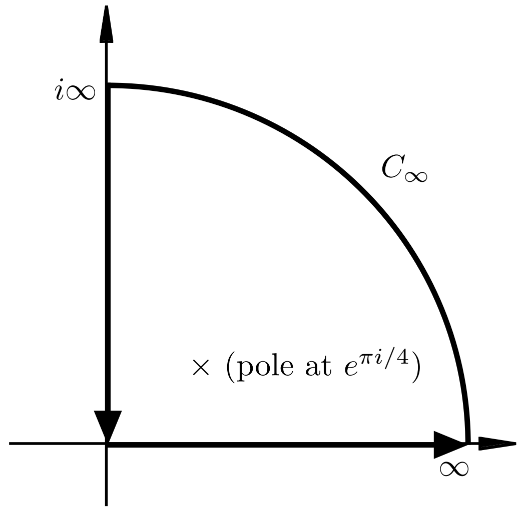
$$f(z) = \int_0^{i\infty} \frac{e^{izv}}{1+(-iv)^4}(-i\,dv) = i\int_{i\infty}^0 \frac{e^{izv}}{1+v^4}\,dv.$$

We now construct the integral along the contour that runs from 0 to ∞ around through the first quadrant to $i\infty$ and then back to zero (see figure).

But by contour integration, we know that

$$\int_0^\infty \frac{e^{izv}}{1+v^4}\,dv + \int_{i\infty}^0 \frac{e^{izv}}{1+v^4}\,dv + \int_{C_\infty} \frac{e^{izv}}{1+v^4}\,dv = 2\pi i(\text{sum of residues}),$$

where C_∞ is the arc portion of the contour at $|v| \to \infty$. But this portion of the contour will vanish whenever $\Im(zv) > 0$, so $0 < \arg z + \arg v < \pi$. But since v is in the first



quadrant, this occurs whenever $0 < \arg z < \pi/2$. Since the only singularity is a simple pole at $v = e^{\pi i/4}$, we have that

$$\begin{aligned} f(z) &= i\left[\frac{2\pi i}{4e^{3\pi i/4}}\exp\left(ize^{\pi i/4}\right) - \int_0^\infty \frac{e^{izv}}{1+v^4}\,dv\right] \\ &= -\frac{\pi e^{-3\pi i/4}}{2}\exp\left(ze^{3\pi i/4}\right) - i\int_0^\infty \frac{e^{izv}}{1+v^4}\,dv \\ &= g(z). \end{aligned}$$

Since f and g agree in this strip, g is the analytic continuation of f. The integral converges whenever $\Im(z) > 0$, so $0 < \arg z < \pi$.

(d) (3 points) Discuss the complete asymptotic expansion for $f(z)$ in $0 < \arg z < \pi$, noting the different behavior in different sectors.

Solution. We now let $z = ix$, where x is real and positive. Substituting this result, we obtain

$$\begin{aligned} g(z) &= \frac{\pi(1+i)}{2\sqrt{2}}\exp\left(ze^{3\pi i/4}\right) - i\int_0^\infty \frac{e^{-xv}}{1+v^4}\,dv \\ &= \frac{\pi(1+i)}{2\sqrt{2}}\exp\left(ze^{3\pi i/4}\right) - i\sum_{n=0}^\infty \frac{(-1)^n\Gamma(4n+1)}{x^{4n+1}} \\ &= \frac{\pi(1+i)}{2\sqrt{2}}\exp\left(ze^{3\pi i/4}\right) - i\sum_{n=0}^\infty \frac{(-1)^n\Gamma(4n+1)}{(-iz)^{4n+1}} \\ &= \frac{\pi(1+i)}{2\sqrt{2}}\exp\left(ze^{3\pi i/4}\right) + \sum_{n=0}^\infty \frac{(-1)^n\Gamma(4n+1)}{z^{4n+1}}. \end{aligned}$$

We note that if $0 < \arg z < 3\pi/4$, the first term is transcendentally small since the real part of the exponent is negative. Thus, it is only for the region $3\pi/4 < \arg z < \pi$ that the residue becomes dominant.