

## Homework Set 7 Solutions

1. (5 points) Find the leading-order asymptotic behavior of solutions to

$$y'' - (\log x)^2 y = 0. \quad (7.1)$$

as  $x \rightarrow \infty$ .

*Solution.* Substituting  $y = e^{S_0}$  into (7.1), we obtain

$$\begin{aligned} e^{S_0} [(S'_0)^2 + S''_0] e^{S_0} - (\log x)^2 e^{S_0} &= 0 \\ (S'_0)^2 + S''_0 &= (\log x)^2. \end{aligned}$$

Since this is in standard form, we assume that  $(S'_0)^2 \gg S''_0$ , so we obtain

$$\begin{aligned} S'_0 &= \pm \log x \\ S_0 &= \pm(x \log x - x), \end{aligned}$$

where we suppress the constant since it is absorbed into the final solution. Now letting  $y = e^{S_0+S_1}$ , we obtain

$$\begin{aligned} e^{S_0+S_1} [(S'_0 + S'_1)^2 + S''_0 + S''_1] e^{S_0+S_1} - (\log x)^2 e^{S_0+S_1} &= 0 \\ \pm \frac{1}{x} \pm 2 \log x S'_1 + (S'_1)^2 + S''_1 &= 0. \end{aligned} \quad (\text{A})$$

Note that  $S''_0 \ll (S'_0)^2$ , as assumed. Now assuming that the first two terms dominate, we obtain

$$\begin{aligned} S'_1 &= -\frac{1}{2x \log x} \\ S_1 &= -\frac{\log(\log x)}{2}, \end{aligned}$$

where we have again absorbed the constant. Note that the first two terms in (A) really do dominate. Therefore, we have that our solutions are of the form

$$\begin{aligned} y &\sim C_{\pm} \exp\left(\pm(x \log x - x) - \frac{\log(\log x)}{2}\right) \\ &\sim \frac{C_{\pm} x^{\pm x} e^{\mp x}}{\sqrt{\log x}}. \end{aligned}$$

2. (9 points) Find the leading-order asymptotic behavior of solutions to

$$y'' = (x + x^\alpha)y, \quad (7.2)$$

as  $x \rightarrow \infty$  and show that the leading-order behavior is exactly that of the Airy functions whenever  $\alpha < -1/2$ .

*Solution.* Substituting  $y = e^{S_0}$  into (7.2), we obtain

$$\begin{aligned} e^{S_0} [(S'_0)^2 + S''_0] e^{S_0} - (x + x^\alpha)e^{S_0} &= 0 \\ (S'_0)^2 + S''_0 &= (x + x^\alpha). \end{aligned}$$

Since this is in standard form, we assume that  $(S'_0)^2 \gg S''_0$ , so we obtain

$$\begin{aligned} S'_0 &= \pm \sqrt{x + x^\alpha} \\ S_0 &= \pm \int^x \sqrt{t + t^\alpha} dt, \end{aligned}$$

where we suppress the constant since it is absorbed into the final solution. Now letting  $y = e^{S_0+S_1}$ , we obtain

$$\begin{aligned} e^{S_0+S_1} [(S'_0 + S'_1)^2 + S''_0 + S''_1] e^{S_0+S_1} - (x + x^\alpha)e^{S_0+S_1} &= 0 \\ \pm \frac{1 + \alpha x^{\alpha-1}}{2\sqrt{x + x^\alpha}} \pm 2S'_1 \sqrt{x + x^\alpha} + (S'_1)^2 + S''_1 &= 0. \end{aligned} \quad (\text{B})$$

Note that  $S''_0 \ll (S'_0)^2$ , as assumed. Now assuming that the first two terms dominate, we obtain

$$\begin{aligned} S'_1 &= -\frac{1 + \alpha x^{\alpha-1}}{4(x + x^\alpha)} \\ S_1 &= -\frac{\log(x + x^\alpha)}{4}, \end{aligned}$$

where we have again absorbed the constant. Note that the first two terms in (B) really do dominate. Therefore, we have that our solutions are of the form

$$\begin{aligned} y &\sim C_\pm \exp\left(\pm \int^x \sqrt{t + t^\alpha} dt - \frac{\log(x + x^\alpha)}{4}\right) \\ &\sim \frac{C_\pm}{(x + x^\alpha)^{1/4}} \exp\left(\pm \int^x \sqrt{t + t^\alpha} dt\right). \end{aligned}$$

Now the Airy functions don't have the  $x^\alpha$  term, so they behave like

$$f(x) \sim \frac{C_\pm}{x^{1/4}} \exp\left(\pm \frac{2x^{3/2}}{3}\right).$$

It should be clear that we won't get this behavior if  $\alpha \geq 1$ . In the other case, we expand the roots using the Binomial Theorem to obtain

$$\begin{aligned} y &\sim \frac{C_{\pm}}{x^{1/4}(1+x^{\alpha-1})^{1/4}} \exp\left(\pm \int^x t^{1/2} \left(1 + \frac{t^{\alpha-1}}{2}\right) dt\right) \\ &\sim \frac{C_{\pm}}{x^{1/4}} \left(1 - \frac{x^{\alpha-1}}{4}\right) \exp\left(\pm \frac{2x^{3/2}}{3} \pm \frac{x^{\alpha+1/2}}{2(\alpha+1/2)}\right) \\ &\sim f(x) \exp\left(\pm \frac{x^{\alpha+1/2}}{2(\alpha+1/2)}\right), \end{aligned}$$

where we have assumed that  $\alpha \neq -1/2$ . We now note that for the behavior to be the same as  $f$ , the argument of the exponential must be  $o(1)$ , which means that  $\alpha < -1/2$ . For the case  $\alpha = -1/2$ , we would have obtained a logarithm as the argument of our exponential, and hence the behavior would be an algebraic power of  $x$  larger (smaller) than the Airy functions.

3. (13 points) Find the general solution of

$$w'' + \left[k^2 + \frac{1}{k(x^2+1)}\right] w = -k^2 \quad (7.3)$$

for large  $k$ . Construct enough terms so that your error is  $o(k^{-3})$ .

*Solution.* We begin by constructing the homogeneous solution. If we work in the standard way by letting

$$w_h(x) = \exp\left(ik\left(\phi_0(x) + k^{-1}\phi_1(x) + \dots\right)\right),$$

the expansions of the various terms in the second derivative will become tedious. However, we know from notes in class that the  $\phi_1(x)$  term typically becomes a coefficient of the exponential. Hence instead of the above we assume a form

$$w_h = \Phi(x; k)e^{ik\phi(x)}.$$

Substituting this form into the homogeneous form of (7.3), we obtain

$$\begin{aligned} \frac{d}{dx} \left( \Phi' e^{ik\phi(x)} + ik\phi' \Phi e^{ik\phi(x)} \right) + \left[ k^2 + \frac{1}{k(x^2+1)} \right] \Phi e^{ik\phi(x)} &= 0 \\ (\Phi'' + 2ik\phi' \Phi' + ik\phi'' \Phi + (ik\phi')^2 \Phi) e^{ik\phi(x)} + \left[ k^2 + \frac{1}{k(x^2+1)} \right] \Phi e^{ik\phi(x)} &= 0 \\ k^2 [1 - (\phi')^2] \Phi + ik(2\phi' \Phi' + \phi'' \Phi) + \Phi'' + \frac{\Phi}{k(x^2+1)} &= 0 \\ \phi' = \pm 1 \quad \implies \quad \pm 2ik\Phi' + \Phi'' + \frac{\Phi}{k(x^2+1)} &= 0. \quad (C) \end{aligned}$$

Then letting

$$\Phi(x; k) = \sum_{j=0}^{\infty} k^{-j} \Phi_j(x)$$

in (C), we obtain, to appropriate orders,

$$\begin{aligned} \pm 2ik(\Phi_0 + k^{-1}\Phi_1 + k^{-2}\Phi_2 + k^{-3}\Phi_3)' + (\Phi_0 + k^{-1}\Phi_1 + k^{-2}\Phi_2)'' + \frac{\Phi_0 + k^{-1}\Phi_1}{k(x^2 + 1)} &= 0 \\ \pm 2ik\Phi_0' = 0 &\implies \Phi_0 = A \\ \pm 2i\Phi_1' = 0 &\implies \Phi_0 = B. \end{aligned}$$

But this gives us two degrees of freedom. How can this be?

Note that the subsequent terms are going to depend on the combination  $\Phi_0 + k^{-1}\Phi_1$  through the fractional term in the equation. Hence our solution will be proportional to  $A + k^{-1}B$ . Hence we may treat this term as a single constant in our expansion. Hence without loss of generality we may set  $A = 1$ ,  $B = 0$  (which is equivalent to dividing our solution by  $A + k^{-1}B$ ). Continuing to simplify, we obtain

$$\begin{aligned} \pm 2ik^{-1}\Phi_2' + \frac{1}{k(x^2 + 1)} &= 0 \\ \Phi_2' &= \pm \frac{i}{2(x^2 + 1)} \implies \Phi_2 = \pm \frac{i}{2} \tan^{-1} x \\ \pm 2ik^{-1}\Phi_3' + k^{-2}\Phi_2'' &= 0 \\ \Phi_3 &= \mp \frac{1}{2i}\Phi_2' = -\frac{1}{2i} \left( \frac{i}{2(x^2 + 1)} \right) = -\frac{1}{4(x^2 + 1)}, \end{aligned}$$

where we can take any constants of integration equal to zero for the same reasons we took  $B = 0$ .

Next we turn our attention to the particular solution. This corresponds to the steady state, so we do not expect rapid oscillations. Hence we let

$$w_p = \sum_{j=0}^{\infty} k^{-j} P_0(x).$$

Substituting this expression into (7.3), we obtain, to appropriate orders,

$$\begin{aligned} (P_0 + k^{-1}P_1)'' + (P_0 + k^{-1}P_1 + k^{-2}P_2 + k^{-3}P_3) \left[ k^2 + \frac{1}{k(x^2 + 1)} \right] &= -k^2 \\ P_0 k^2 = -k^2 &\implies P_0 = -1 \\ P_1 k = 0, \quad P_2 = 0, \quad k^{-1} \left( P_3 + \frac{P_0}{x^2 + 1} \right) = 0 &\implies P_3 = -\frac{1}{x^2 + 1}. \end{aligned}$$

Therefore, our solution is given by

$$\begin{aligned} w(x) &= w_h(x) + w_p(x) \\ &= A_+ e^{ikx} \left( 1 + \frac{i}{2k^2} \tan^{-1} x - \frac{1}{4k^3(x^2 + 1)} \right) \\ &\quad + A_- e^{-ikx} \left( 1 - \frac{i}{2k^2} \tan^{-1} x - \frac{1}{4k^3(x^2 + 1)} \right) - \left[ 1 + \frac{1}{k^3(x^2 + 1)} \right]. \end{aligned}$$

4. (13 points) Consider the system

$$x^2 y'' \sqrt{2} = [(x+1)y + xz], \quad (7.4a)$$

$$x^2 z'' \sqrt{2} = -[xy + (x+1)z], \quad (7.4b)$$

Show that

$$\begin{aligned} y(x) &\sim \left( \frac{x^{5/8}}{\sqrt{2}} + \frac{x^{1/8}}{2} \right) e^{S_0(x)} \\ z(x) &\sim \left( -\frac{x^{5/8}}{\sqrt{2}} + \frac{x^{1/8}}{2} \right) e^{S_0(x)}. \end{aligned}$$

as  $x \rightarrow \infty$ , and determine  $S_0(x)$ . Why is it not appropriate to naively approximate the  $x+1$  terms in (7.4) by  $x$ ?

*Solution.* If we take the sum and difference of (7.4), we obtain

$$x^2 u'' \sqrt{2} = v, \quad (D.1)$$

$$x^2 v'' \sqrt{2} = (2x+1)u, \quad (D.2)$$

where

$$u = y + z, \quad v = y - z.$$

Then combining equations (D), we have

$$\begin{aligned} x^2 \left( x^2 u'' \sqrt{2} \right)'' \sqrt{2} &= (2x+1)u \\ x^2 u^{(4)} + 2xu^{(3)} + 2u'' - \frac{2x+1}{2x^2} u &= 0. \end{aligned} \quad (E)$$

Substituting  $u = e^{S_0}$  into (E), we see that the fourth derivative will yield a large number of terms. Each of these will be a product of derivatives, where the sum of the orders is equal to 4. Following the reasoning for second-order problems, we assume that  $S_0'$  is a larger term. If we do that, then the dominant two terms are going to be

$$\begin{aligned} y' &= S_0' e^{S_0}, \\ y'' &= [(S_0')^2 + S_0''] e^{S_0}, \\ y^{(3)} &\sim [(S_0')^3 + 3S_0' S_0''] e^{S_0}, \\ y^{(4)} &\sim [(S_0')^4 + 6(S_0')^2 S_0''] e^{S_0}. \end{aligned} \quad (F.1)$$

$$y^{(4)} \sim [(S_0')^4 + 6(S_0')^2 S_0''] e^{S_0}. \quad (F.2)$$

$(S'_0)^4$  is the largest term. Then (E) becomes, to leading order,

$$\begin{aligned} x^2(S'_0)^4 e^{S_0} - \frac{1}{x} e^{S_0} &= 0 \\ S'_0 &= (x^{-3})^{1/4} = \frac{\omega}{x^{3/4}}, \quad \omega^4 = 1, \\ S_0 &= 4\omega x^{1/4}. \end{aligned}$$

Since  $S_0^{(j)} \propto x^{1/4-j}$ , we see that the  $j$ th power of  $S'_0$  is much larger than the the  $j$ th derivative of  $S_0$ ,  $j > 1$ . Therefore, our assumption checks.

Now we let  $y = e^{S_0+S_1}$ . Following the previous reasoning, we see that the next-order terms we should consider would be

$$x(S'_0)^3 = O(x^{-5/4}) \text{ (F.1)}, \quad x^2(S'_0)^2 S''_0 = O(x^{-5/4}) \text{ (F.1)}, \quad x^2(S'_0)^3 S'_1 \text{ (F.2)}.$$

Note that these terms are all larger than the  $-u/2x^2$  term in (E). Extracting these terms from (E) (with their proper coefficients), we have

$$\begin{aligned} x^2 [4(S'_0)^3 S'_1 + 6(S'_0)^2 S''_0] e^{S_0+S_1} + 2x(S'_0)^3 e^{S_0+S_1} &= 0 \\ x \left[ 4 \frac{\omega^3}{x^{9/4}} S'_1 + 6 \frac{\omega^2}{x^{3/2}} \left( -\frac{3\omega}{4x^{7/4}} \right) \right] + 2 \frac{\omega^3}{x^{9/4}} &= 0 \\ 4xS'_1 - \frac{9}{2} + 2 &= 0. \\ S'_1 &= \frac{1}{8x} \\ S_1 &= \frac{\log x}{8} \\ e^{S_1} &= x^{1/8} \\ u(x) &\sim x^{1/8} \exp(4\omega x^{1/4}). \end{aligned}$$

Then using (D.1), we have

$$\begin{aligned} v(x) &\sim x^2 \sqrt{2} \frac{d^2}{dx^2} \left[ x^{1/8} \exp(4\omega x^{1/4}) \right] \sim x^2 (x^{-3/4})^2 x^{1/8} \sqrt{2} \exp(4\omega x^{1/4}) \\ &\sim x^{5/8} \sqrt{2} \exp(4\omega x^{1/4}) \\ y(x) &= \frac{u+v}{2} \sim \frac{x^{1/8} + x^{5/8} \sqrt{2}}{2} \exp(4\omega x^{1/4}), \\ z(x) &= \frac{u-v}{2} \sim \frac{x^{1/8} - x^{5/8} \sqrt{2}}{2} \exp(4\omega x^{1/4}). \end{aligned}$$

With some reworking, we have the desired result.

Note that if we had replaced  $x+1$  with  $x$  in our original system, (D.1) would have become

$$x^2 u'' \sqrt{2} = 0,$$

which would have yielded the wrong behavior for  $u$  (and hence  $y$  and  $z$ ).