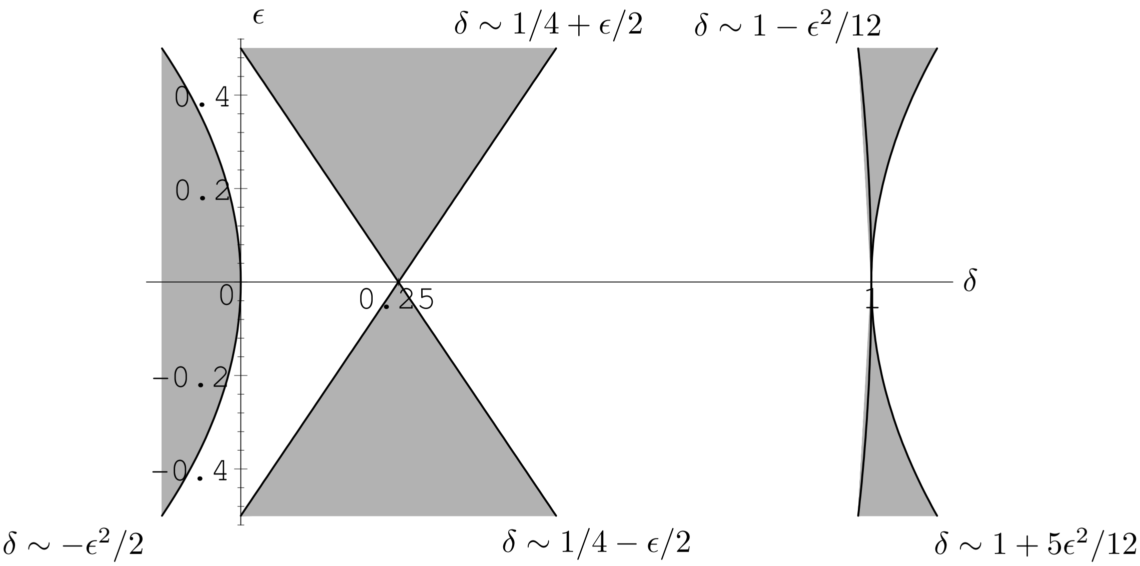
Homework Set 6 Solutions



Shaded regions are unstable.

1. Above is shown a graph of the (δ, ϵ) plane for the standard Mathieu equation. Now consider the Mathieu equation with an extra term added:

$$u'' + 2\beta u' + (\delta + \epsilon \cos t)u = 0, \qquad |\epsilon| \ll 1, \tag{6.1}$$

where β is a constant. We wish to see how the stability diagram for $\beta = 0$ changes when $\beta \neq 0$.

(a) (5 points) By introducing a new variable v, one can reduce the equation (6.1) for u to the standard Mathieu equation for v (perhaps with different parameters). Determine the relationship between u and v.

Solution. The β term should introduce exponential growth or decay, so we let $u = e^{\alpha t} v$ in (6.1) to obtain

$$\alpha^2 e^{\alpha t} v + 2\alpha e^{\alpha t} v' + e^{\alpha t} v'' + 2\beta \left(\alpha e^{\alpha t} v + e^{\alpha t} v'\right) + (\delta + \epsilon \cos t) e^{\alpha t} v = 0$$
$$v'' + 2(\beta + \alpha) v' + (\alpha^2 + 2\alpha\beta + \delta + \epsilon \cos t) v = 0.$$

Therefore, by setting $\alpha = -\beta$, we obtain

$$v'' + (\delta - \beta^2 + \epsilon \cos t) v = 0. \tag{A}$$

(b) (9 points) Explain why if β is "large," the solution will be dominated by the linear damping/growth term. Determine "large" for each of the special values $n^2/4$ of δ_0.

Solution. Since $u = e^{-\beta t}v$, we see that the $e^{-\beta t}$ term will dominate unless βt is on the order of the slow-time scales established for the standard Mathieu equation. Therefore, for $n > 1$, any $\beta > O(\epsilon^2)$ will cause the linear term to dominate, and for $n \leq 1$, any $\beta > O(\epsilon)$ will cause the linear term to dominate.

(c) (17 points) Determine the new stability curves for the cases $n = 0$, $n = 1$, and $n = 2$ when the linear and nonlinear effects balance [*i.e.*, when β is at the threshold values from part (b).]

Solution. We begin with the case $n = 1$ and let $\beta = \beta_1\epsilon$. We note that in this case, the β^2 term in (A) is $O(\epsilon^2)$, which is beyond the number of terms used to obtain the solution. Thus, using our previous results, we see that the slow-time amplitude is now given by $e^{\gamma\tau}$, where

$$\tau = \epsilon t, \qquad \gamma = -\beta_1 \pm \sqrt{\frac{1}{4} - \delta_1^2}.$$

We note immediately that we can have a stable solution only if $\beta_1 > 0$. Therefore, all solutions with $\beta_1 < 0$ are unstable with weak forcing. If $\beta_1 > 0$ and γ is real, then $\gamma \leq 0$ whenever

$$\delta_1^2 \geq \frac{1}{4} - \beta_1^2, \tag{B.1}$$

with the equality sign giving the transition curve. Then using our definitions of δ_1 and β_1, we have

$$\left(\frac{\delta - 1/4}{\epsilon}\right)^2 \geq \frac{1}{4} - \left(\frac{\beta}{\epsilon}\right)^2$$

$$\left(\delta - \frac{1}{4}\right)^2 - \frac{\epsilon^2}{4} \geq \beta^2. \tag{B.2}$$

Now we proceed to the case $n = 2$ and let $\beta = \beta_2\epsilon^2$. We note that in this case, the β^2 term in (A) is $O(\epsilon^4)$, which is beyond the number of terms used to obtain the solution. Thus, using our previous results, we see that the slow-time amplitude is now given by $e^{\gamma\tau}$, where

$$\gamma = -\beta_2 \pm \frac{1}{2}\sqrt{\left(\delta_2 + \frac{1}{12}\right)\left(\frac{5}{12} - \delta_2\right)}.$$

We note immediately that we can have a stable solution only if $\beta_2 > 0$. Therefore, all solutions with $\beta_2 < 0$ are unstable with weak forcing. If $\beta_1 > 0$ and γ is real, then $\gamma \leq 0$ whenever

$$\beta_2^2 \geq \frac{1}{4}\left(\delta_2 + \frac{1}{12}\right)\left(\frac{5}{12} - \delta_2\right), \tag{C.1}$$

with the equality sign giving the transition curve. Then using our definitions of δ_2 and β_2, we have

$$\left(\frac{\beta}{\epsilon^2}\right)^2 \geq \left(\frac{\delta-1}{\epsilon^2}+\frac{1}{12}\right)\left(\frac{5}{12}-\frac{\delta-1}{\epsilon^2}\right)$$

$$\left(\delta-1+\frac{\epsilon^2}{12}\right)\left(\delta-1-\frac{5\epsilon^2}{12}\right) \geq \beta^2. \tag{C.2}$$

Lastly, we consider the case $n = 0$. Since suppression of secularity took place at the equation *two* orders beyond the leading order, we see that the β^2 term *does* contribute. In particular, the $O(\epsilon)$ equation becomes

$$\begin{aligned}\frac{\partial^2 F_1}{\partial t^2} &= -\cos t\,[B_0(\tau)+B_{-1}(\tau)\cos t] - B''_{-1}(\tau) - (\delta_2-\beta_1^2)B_{-1}(\tau)\\ &= -\left[B''_{-1}(\tau)+(\delta_2+1/2-\beta_1^2)B_{-1}(\tau)\right] + \text{ higher harmonics.}\end{aligned}$$

To suppress the singularity, we must solve the following differential equation:

$$B''_{-1}(\tau)+(\delta_2+1/2-\beta_1^2)B_{-1}(\tau)=0$$

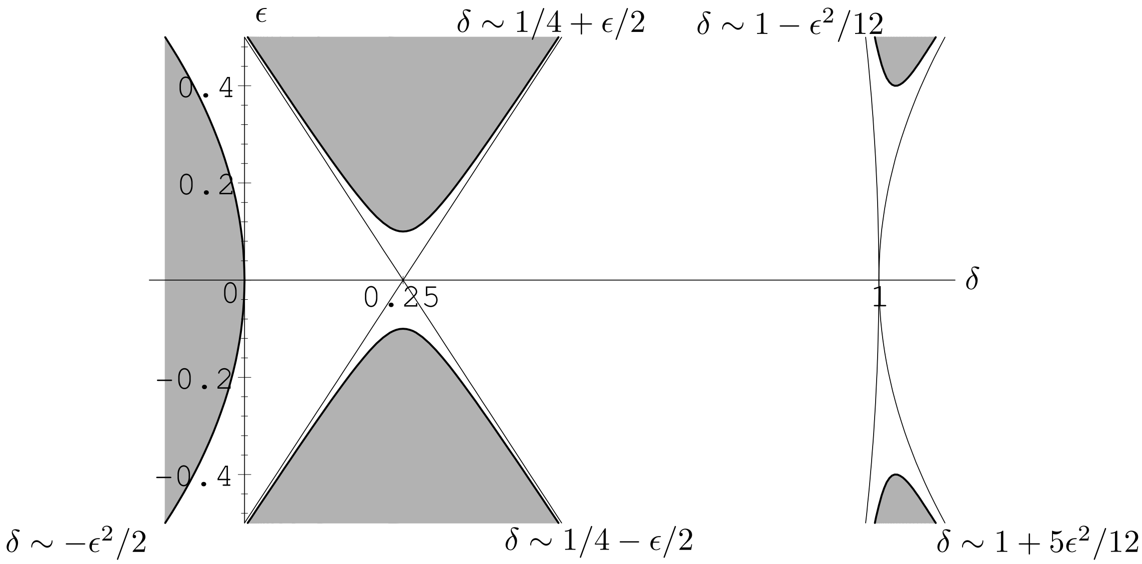
to get the behavior in v. To obtain the behavior in u, we see that the slow-time amplitude is now given by $e^{\gamma\tau}$, where

$$\gamma = -\beta_1 \pm \sqrt{\beta_1^2-\delta_2-\frac{1}{2}}.$$

We note immediately that we can have a stable solution only if $\beta_1 > 0$. Therefore, all solutions with $\beta_1 < 0$ are unstable with weak forcing. If $\beta_1 > 0$ and γ is real, then $\gamma \leq 0$ whenever $\beta_1 \geq 0$, so this curve does not change at all.

(d) (9 points) Sketch the new stability diagram.

Solution. We note that for any $\beta \neq 0$, (B.2) is the graph of a hyperbola in the δ-ϵ plane with asymptotes given by the curves with $\beta = 0$. Similarly, (C.2) also asymptotes to the curves with $\beta = 0$. Therefore, our solution diagram is given by the following:



Shaded regions are unstable; $\beta_1 = 0.05$, $\beta_2 = 0.01$.