Homework Set 5 Solutions

1. Consider the following problem:

$$\ddot{u} + u + \epsilon u^2 = 0, \qquad 0 < \epsilon \ll 1, \qquad u(0) = \alpha, \qquad \dot{u}(0) = 0.$$

(a) (4 points) At what order of ϵ do secular-causing terms first appear? You need not work through all the details.

Solution. (*Here more detail is shown than was necessary for full credit.*) Letting $u(t;\epsilon) \sim u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots$, we have

$$\frac{d^2}{dt^2}(u_0 + \epsilon u_1 + \epsilon^2 u_2) + (u_0 + \epsilon u_1 + \epsilon^2 u_2) + \epsilon(u_0 + \epsilon u_1)^2 = 0$$

$$\begin{aligned}
\ddot{u}_0 + u_0 &= 0, & u_0(0) &= \alpha, & \dot{u}_0(0) &= 0 & & O(1)\\
\ddot{u}_1 + u_1 + u_0^2 &= 0, & u_1(0) &= 0, & \dot{u}_1(0) &= 0 & & O(\epsilon)\\
\ddot{u}_2 + u_2 + 2u_0u_1 &= 0, & u_2(0) &= 0, & \dot{u}_2(0) &= 0 & & O(\epsilon^2)
\end{aligned}$$

Solving our equations in turn, we have

$$\begin{aligned}
u_0 &= \alpha\cos t,\\
\ddot{u}_1 + u_1 &= -\alpha^2\cos^2 t = -\frac{\alpha^2}{2} - \frac{\alpha^2\cos 2t}{2} \qquad \text{(A)}\\
u_1 &= -\frac{\alpha^2}{2} + \frac{\alpha^2\cos 2t}{6} + A\cos t + B\sin t\\
&= -\frac{\alpha^2}{2} + \frac{\alpha^2\cos 2t}{6} + \frac{\alpha^2\cos t}{3},\\
\ddot{u}_2 + u_2 &= \alpha^3\cos t\left[1 - \frac{\cos 2t}{3} - \frac{2\cos t}{3}\right]\\
&= \frac{5\alpha^3}{6}\cos t + \text{acceptable terms}
\end{aligned}$$

So we see that the first secular-causing term occurs at $O(\epsilon^2)$. This happens because in (A) we see that squaring a trigonometric term does not cause a problem. (The solution would still be okay at $O(\epsilon)$ if we had both sin and cos terms.)

(b) (11 points) Show that the $O(1)$ uniformly valid approximation for this problem is

$$F_0(T,\tau) = \alpha\cos\left(T - \frac{5\alpha^2\tau}{12}\right)$$

for appropriately chosen T and τ.

Solution. We see from (a) that our choice of τ should be $\tau = \epsilon^2 t$. As in class, we let

$$y(t;\epsilon) = F_0(T,\tau) + \epsilon F_1(T,\tau) + \epsilon^2 F_2(T,\tau) + o(\epsilon^2), \qquad T = t\left[1 + \omega_1\epsilon + o(\epsilon^2)\right],$$

$$\frac{d}{dt} = (1+\omega_1\epsilon)\frac{\partial}{\partial T} + \epsilon^2\frac{\partial}{\partial \tau} + o(\epsilon^2).$$

Substituting this expression into (1), we have, to leading orders,

$$(1+\omega_1\epsilon)\frac{\partial}{\partial T}\left[(1+\omega_1\epsilon)\frac{\partial}{\partial T}(F_0 + \epsilon F_1 + \epsilon^2 F_2) + \epsilon^2\frac{\partial F_0}{\partial \tau}\right] + \epsilon^2\frac{\partial}{\partial \tau}\left(\frac{\partial F_0}{\partial T}\right) + F_0 + \epsilon F_1 + \epsilon^2 F_2 + \epsilon(F_0 + \epsilon F_1)^2 = 0.$$

Expanding and taking only the terms to $O(\epsilon^2)$, we have

$$\frac{\partial^2 F_0}{\partial T^2} + F_0 = 0, \qquad O(1)$$

$$\frac{\partial^2 F_1}{\partial T^2} + F_1 + 2\omega_1\frac{\partial^2 F_0}{\partial T^2} + F_0^2 = 0, \qquad O(\epsilon):\text{(B)}$$

$$\frac{\partial^2 F_2}{\partial T^2} + F_2 + 2\frac{\partial^2 F_0}{\partial T\partial \tau} + 2\omega_1\frac{\partial^2 F_1}{\partial T^2} + 2F_0F_1 + \omega_1^2\frac{\partial^2 F_0}{\partial T^2} = 0. \qquad O(\epsilon^2)$$

We need only the leading order of our boundary conditions:

$$F_0(0,0) = \alpha, \qquad \frac{\partial F_0}{\partial T}(0,0) = 0.$$

Now we solve our equations one at a time:

$$F_0 = A_0(\tau)\cos T + B_0(\tau)\sin T, \qquad A_0(0) = \alpha, \qquad B_0(0) = 0.$$

$$\frac{\partial^2 F_1}{\partial T^2} + F_1 = -2\omega_1\left(A_0\cos T + B_0\sin T\right) - \left(A_0\cos T + B_0\sin T\right)^2$$

$$\frac{\partial^2 F_1}{\partial T^2} + F_1 = -2\omega_1\left(A_0\cos T + B_0\sin T\right) - \frac{A_0^2 + B_0^2}{2} - A_0B_0\sin 2T - \frac{(A_0^2 - B_0^2)\cos 2T}{2}.$$

We see that in order to suppress secularity we must have $\omega_1 = 0$, and then we have

$$F_1 = -\frac{A_0^2 + B_0^2}{2} + \frac{A_0B_0\sin 2T}{3} + \frac{(A_0^2 - B_0^2)\cos 2T}{6} + A_1(\tau)\cos T + B_1(\tau)\sin T. \qquad \text{(C)}$$

Going to the next order and recalling that $\omega_1 = 0$, we have

$$\frac{\partial^2 F_2}{\partial T^2} + F_2 = -2\frac{\partial^2 F_0}{\partial T \partial \tau} - 2F_0F_1 \tag{D}$$

$$\begin{aligned}\frac{\partial^2 F_2}{\partial T^2} + F_2 &= -2\left(B_0'\cos T - A_0'\sin T\right) - 2\left(A_0\cos T + B_0\sin T\right)\times \\ &\qquad \left[-\frac{A_0^2+B_0^2}{2} + \frac{A_0B_0\sin 2T}{3} + \frac{(A_0^2-B_0^2)\cos 2T}{6}\right] + \text{acceptable terms} \\ &= 2\left[-B_0' + \frac{A_0(A_0^2+B_0^2)}{2} - \frac{A_0(A_0^2-B_0^2)}{12} - \frac{A_0B_0^2}{6}\right]\cos T \\ &\quad + 2\left[A_0' + \frac{B_0(A_0^2+B_0^2)}{2} + \frac{B_0(A_0^2-B_0^2)}{12} - \frac{A_0^2B_0}{6}\right]\sin T \\ &\quad + \text{acceptable terms.}\end{aligned} \tag{E}$$

Therefore, we see that both bracketed terms must be set equal to zero to suppress secularity:

$$B_0' = \frac{5A_0(A_0^2+B_0^2)}{12} \tag{F.1}$$

$$A_0' = -\frac{5B_0(A_0^2+B_0^2)}{12}. \tag{F.2}$$

Combining equations (F), we see that we have

$$A_0A_0' + B_0B_0' = (A_0^2+B_0^2)' = 0 \qquad \Longrightarrow \qquad A_0^2 + B_0^2 = \alpha.$$

Therefore, we may write

$$A_0(\tau) = \alpha\cos\phi_0(\tau), \qquad B_0(\tau) = \alpha\sin\phi_0(\tau), \qquad \phi_0(0) = 0,$$

in which case (F.1) becomes

$$\begin{aligned}\alpha\phi_0'\cos\phi_0 &= \frac{5\alpha^3\cos\phi_0}{12} \\ \phi_0' &= \frac{5\alpha^2}{12} \\ \phi_0(\tau) &= \frac{5\alpha^2\tau}{12},\end{aligned}$$

so

$$\begin{aligned}F_0(T,\tau) &= \alpha\cos\frac{5\alpha^2\tau}{12}\cos T + \alpha\sin\frac{5\alpha^2\tau}{12}\sin T \\ &= \alpha\cos\left(T - \frac{5\alpha^2\tau}{12}\right),\end{aligned}$$

as required.

2. (25 points) For the following problem:

$$\ddot{u} + u - \epsilon u^2 = \epsilon \cos t, \qquad 0 < \epsilon \ll 1, \tag{5.1}$$

calculate the following:

(a) the proper expansion for $u(t;\epsilon)$,
(b) the proper slow-time scale τ, and
(c) the proper evolution equations for $A_0(\tau)$ and $B_0(\tau)$.

Solution. The key to this problem is to balance the effects of the nonlinearity on the left-hand side with the forcing on the right-hand side. It should be clear that a regular perturbation expansion of the form

$$u = u_0 + \epsilon u_1 + \cdots$$

will fail at $O(\epsilon)$ due to the forcing on the right-hand side. This might lead one to conclude that

$$u(t;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n F_n(T,\tau), \qquad \tau = \epsilon t, \qquad T = t\left(1 + \sum_{n=2}^{\infty} \epsilon^n \omega_n\right).$$

As in class, we let

$$y(t;\epsilon) = F_0(T,\tau) + \epsilon F_1(T,\tau) + o(\epsilon), \qquad T = t\left[1 + O(\epsilon^2)\right],$$

$$\frac{d}{dt} = \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau} + O(\epsilon^2).$$

Substituting these expressions into (5.1), we have, to leading orders,

$$\frac{\partial}{\partial T}\left[\frac{\partial}{\partial T}(F_0 + \epsilon F_1) + \epsilon \frac{\partial F_0}{\partial \tau}\right] + \epsilon \frac{\partial}{\partial \tau}\left(\frac{\partial F_0}{\partial T}\right) + F_0 + \epsilon F_1 + \epsilon^2 F_2 - \epsilon(F_0 + \epsilon F_1)^2 = \epsilon \cos T.$$

Expanding and taking only the terms to $O(\epsilon)$, we have

$$\frac{\partial^2 F_0}{\partial T^2} + F_0 = 0, \qquad O(1)$$

$$\frac{\partial^2 F_1}{\partial T^2} + 2\frac{\partial^2 F_0}{\partial T \partial \tau} + F_1 - F_0^2 = \cos T, \qquad O(\epsilon)$$

Solving our equations one at a time, we have

$$F_0 = A_0(\tau)\cos T + B_0(\tau)\sin T,$$

$$\frac{\partial^2 F_1}{\partial T^2} + F_1 = (A_0 \cos T + B_0 \sin T)^2 + \cos T - 2(-A_0' \sin T + B_0' \cos T)$$

$$\frac{\partial^2 F_1}{\partial T^2} + F_1 = (1 - 2B_0')\cos T + \frac{A_0^2 + B_0^2}{2} + A_0 B_0 \sin 2T + \frac{(A_0^2 - B_0^2)\cos 2T}{2} + 2A_0' \sin T.$$

Therefore, we see that we have no way to suppress secularity, since to do so would make B_0 blow up as $\tau \to \infty$.

Hence we try to scale the problem. Since B_0 blew up in our first try, we expect large oscillations, so we let $u = \epsilon^{-\alpha} y$, where we expect $\alpha > 0$:

$$\epsilon^{-\alpha}\ddot{y} + \epsilon^{-\alpha}y - \epsilon^{1-2\alpha}y^2 = \epsilon \cos t$$
$$\ddot{y} + y - \epsilon^{1-\alpha}y^2 = \epsilon^{1+\alpha}\cos t.$$

We now closely examine the operator on the left-hand side. Since $\alpha < 0$, $\epsilon^{1-\alpha} \ll 1$. Our work in problem 1 convinces us that if we have an equation

$$\ddot{x} + x \pm \delta x^2 = 0,$$

where $0 < \delta \ll 1$, an expansion that allows the nonlinearity to suppress any secularity is given by

$$x(t;\delta) = \sum_{n=0}^{\infty} \delta^n F_n(T,\tau), \qquad \tau = \delta^2 t, \qquad T = t\left(1 + \sum_{n=3}^{\infty} \delta^n \omega_n\right).$$

Note that we have used the fact that we derived that $\omega_1 = 0$. This expansion has terms which allow one to suppress secularity in the equation for F_2 at $O(\delta^2)$. The equation for F_1 at $O(\delta)$ has no secular terms.

What we wish to do now is to ensure that the forcing term on the right-hand side appears where the expansion has terms which can suppress it, namely at $O(\delta^2)$. Of course, in our problem, $\delta = \epsilon^{1-\alpha}$, so we have that

$$\epsilon^{2-2\alpha} = \epsilon^{1+\alpha} \qquad \Longrightarrow \qquad \alpha = 1/3.$$

Therefore, we see that $\delta = \epsilon^{2/3}$ and the correct expansion is given by

$$u(t;\epsilon) = \epsilon^{-1/3}\sum_{n=0}^{\infty} \epsilon^{2n/3} F_n(T,\tau), \qquad \tau = \epsilon^{4/3}t, \qquad T = t\left(1 + \sum_{n=3}^{\infty} \epsilon^{2n/3}\omega_n\right).$$

Now the problem is like problem 1 in that the equations at each order are similar (off by simply a minus sign in the nonlinear term). Therefore, we immediately see that we have

$$F_0 = A_0(\tau)\cos T + B_0(\tau)\sin T, \qquad O(\epsilon^{-1/3})$$
$$\frac{\partial^2 F_1}{\partial T^2} + F_1 = F_0^2. \qquad O(\epsilon^{1/3})$$

Since the forcing is exactly the negative of that in (B), we see that our particular solution for F_1 should be the negative of (C):

$$F_1 = \frac{A_0^2 + B_0^2}{2} - \frac{A_0 B_0 \sin 2T}{3} - \frac{(A_0^2 - B_0^2)\cos 2T}{6} + A_1(\tau)\cos T + B_1(\tau)\sin T. \quad O(\epsilon^{1/3})$$

$$\frac{\partial^2 F_2}{\partial T^2} + F_2 = \cos T - 2\frac{\partial^2 F_0}{\partial T \partial \tau} + 2F_0F_1. \qquad O(\epsilon)$$

We note the similarity between the above equation and (D). But the particular solution for F_1 (which is the only part that contributes to the secularity) is negative the result in problem 1, so the two minus signs cancel. Hence, equation (E) holds with the addition of the $\cos T$ term, and so we have

$$\begin{aligned}\frac{\partial^2 F_2}{\partial T^2} + F_2 = 2\left[\frac{1}{2} - B_0' + \frac{A_0(A_0^2+B_0^2)}{2} - \frac{A_0(A_0^2-B_0^2)}{12} - \frac{A_0B_0^2}{6}\right]\cos T \\ + 2\left[A_0' + \frac{B_0(A_0^2+B_0^2)}{2} + \frac{B_0(A_0^2-B_0^2)}{12} - \frac{A_0^2B_0}{6}\right]\sin T \\ + \text{acceptable terms,}\end{aligned}$$

and equations (F) from the key to HW 5 become

$$B_0' = \frac{1}{2} + \frac{5A_0(A_0^2+B_0^2)}{12} \qquad \text{(G.1)}$$

$$A_0' = -\frac{5B_0(A_0^2+B_0^2)}{12}. \qquad \text{(G.2)}$$

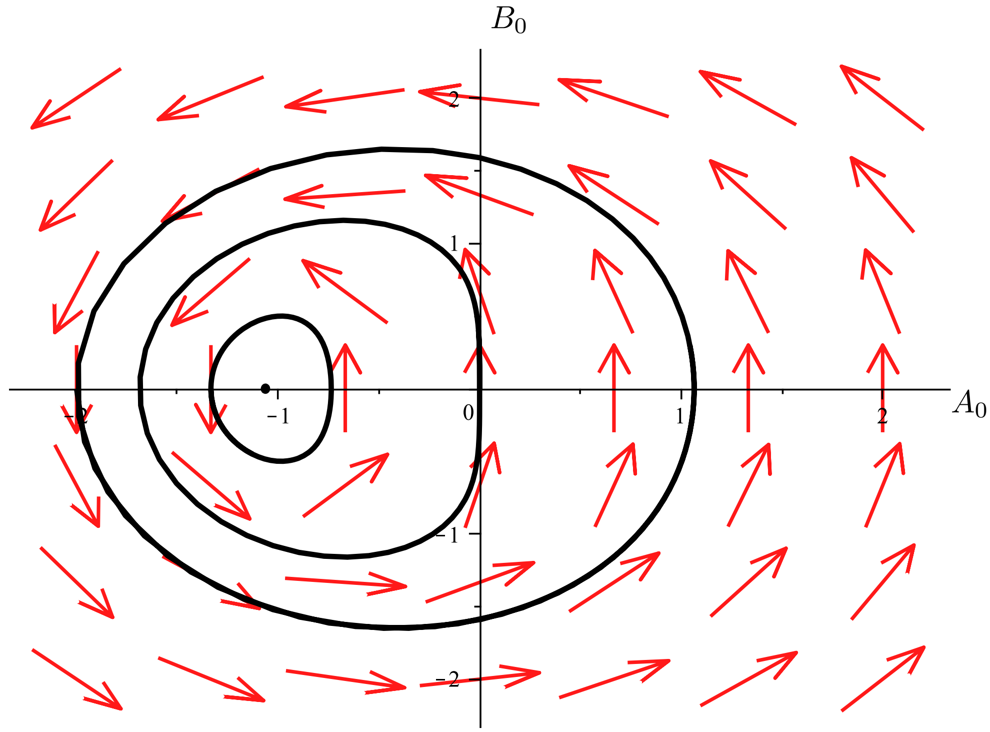
Equations (G) cannot be easily solved in closed form; the phase plane is given below. The coordinates of the center are given where both A_0' and B_0' are zero, so we have

$$\begin{aligned}-\frac{1}{2} &= \frac{5A_0(A_0^2+B_0^2)}{12} \\ 0 &= \frac{5B_0(A_0^2+B_0^2)}{12}.\end{aligned}$$

Hence the solutions

$$A_0(\tau) \equiv -\left(\frac{6}{5}\right)^{1/3}, \qquad B_0(\tau) \equiv 0,$$

are steady-states, as suggested in the remarks. The phase plane is shown below. Note that even when we start at the origin, we obtain a large oscillation.



Phase portrait for (G).