

Homework Set 4 Solutions

1. Consider the following singular boundary-value problem:

$$\epsilon y'' + \frac{y'}{x} - 2y = 0, \quad \epsilon^2 \leq x \leq 1, \quad 0 < \epsilon \ll 1, \quad (4.1a)$$

$$y(\epsilon^2) = 1, \quad y(1) = 1. \quad (4.1b)$$

(a) (3 points) Construct any needed leading-order outer expansions.

Solution. Letting $y \sim y_0 + o(1)$ and letting $\epsilon \rightarrow 0$, we have

$$\begin{aligned} \frac{y_0'}{x} - 2y_0 &= 0, & y_0(0) &= 1, & y_0(1) &= 1, \\ y_0 &= Ae^{x^2}, \\ y_0 &= e^{x^2-1}, \end{aligned}$$

where we have satisfied the right-hand boundary because it isn't the singular one.

(b) (5 points) Construct any needed leading-order inner expansions.

Solution. Inserting the obvious scalings, we obtain

$$\xi = \frac{x}{\epsilon^2}, \quad y(x) \sim w(\xi),$$

$$\begin{aligned} \epsilon^{-3}w'' + \epsilon^{-4}\frac{w'}{\xi} - 2w &= 0 \\ \epsilon w'' + \frac{w'}{\xi} - 2\epsilon^4 w &= 0, & w(1) &= 1, & w(\infty) &= e^{-1}. \end{aligned} \quad (A)$$

The leading order solution is a constant which can't satisfy the boundary conditions. Therefore, we introduce yet another boundary layer:

$$\zeta = \frac{\xi - 1}{\epsilon^a}, \quad w(\xi) \sim f(\zeta).$$

Using these scalings, equations (A) become, to leading order,

$$\begin{aligned} \epsilon^{1-2a}f'' + \epsilon^{-a}\frac{f'}{1+\epsilon^a\zeta} - 2\epsilon^4 f &= 0 & \implies & & a &= 1 \\ f'' + f' &= 0, & f(1) &= 1, & f(\infty) &= e^{-1}, \\ f(\zeta) &= e^{-1} + (1 - e^{-1})e^{-\zeta}. \end{aligned}$$

- (c) (4 points) Construct and sketch the leading-order uniformly valid approximation of the solution(s).

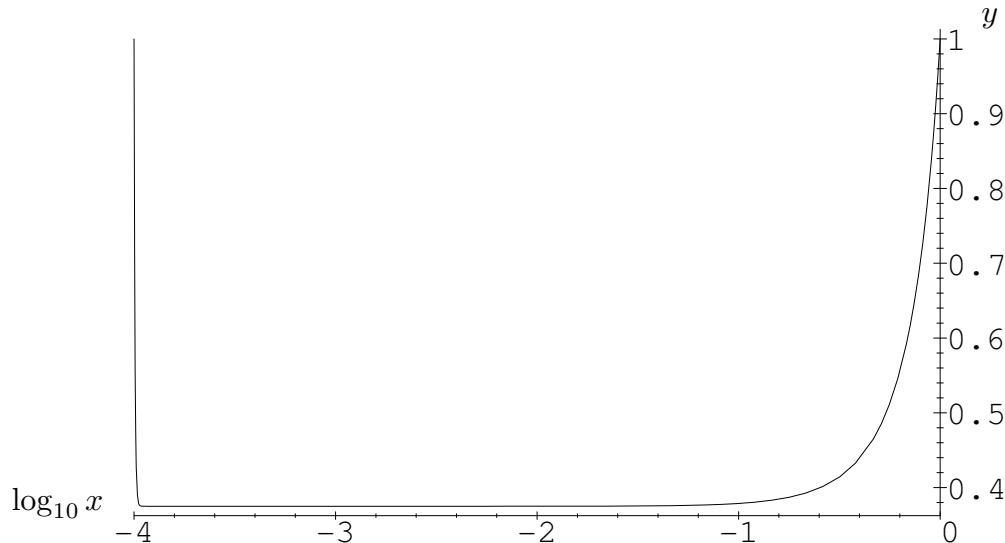
Solution. Since $a = 1$, we have that

$$\zeta = \frac{\xi - 1}{\epsilon} = \frac{x - \epsilon^2}{\epsilon^3}.$$

Writing the inner solution in terms of x , adding it to the outer solution, and subtracting the common part, which is e^{-1} , we have

$$y_u = e^{x^2-1} + (1 - e^{-1}) \exp\left(\frac{\epsilon^2 - x}{\epsilon^3}\right).$$

Here's a graph with $\epsilon = 0.01$.



2. Consider the following system:

$$\epsilon v'' + u'v' = 0, \quad u(0) = u(1) = 0, \quad (4.2)$$

$$u'' = v, \quad v(0) = 2v_* > 0, \quad v(1) = 2\alpha^2 v_*, \quad \alpha > 1. \quad (4.3)$$

- (a) (6 points) Write down the proper scalings and equations for each needed layer. Explain why your choice of layers is unique.

Solution. We begin by noting from (4.3) that if u changes by an $O(1)$ quantity in a boundary layer of width ϵ^a , v will change by an $O(\epsilon^{-2a})$ amount. Therefore, our guiding principle will be to satisfy the conditions on u first, then worry about v . We begin by writing $u = u_0 + o(1)$, $v = v_0 + o(1)$, in which case (4.2) becomes

$$u'_0 v'_0 = 0.$$

Always satisfying the u boundary conditions, we see that $u'_0 = 0$ yields $u_0 \equiv 0$. $v'_0 = 0$ implies that v is a constant, which doesn't satisfy both boundary conditions on v . A constant v implies a parabolic profile for u . The most that we can do is satisfy three of the boundary conditions, which we can do with either of the two solutions:

$$\begin{aligned} v_{\text{I}}(x) = 2v_* & & u_{\text{I}}(x) = v_*x(x-1) & & u'_{\text{I}}(1) = v_* > 0 \\ v_{\text{II}}(x) = 2\alpha^2v_* & \implies & u_{\text{II}}(x) = \alpha^2v_*x(x-1) & \implies & u'_{\text{II}}(0) = -\alpha^2v_* < 0 \end{aligned} \quad (\text{B})$$

Scaling (4.2) by

$$\xi = \frac{x-l}{\epsilon^b}, \quad v(x) = w(\xi), \quad (\text{C.1})$$

we have

$$\epsilon^{1-2b}w'' + \epsilon^{-b}u'_0(l)w' = 0, \quad u'_0(l) \neq 0. \quad (\text{C.2})$$

But we note that if we scale near $x = 0$, the solutions grow exponentially since $u'_{\text{II}}(0) < 0$. Similarly, if we scale near $x = 1$, the solutions grow exponentially since $u'_{\text{I}}(1) > 0$.

Therefore, we must scale about some interior point. But in that case we would still have exponentially growing solutions to (C.2). Hence we conclude that $u'_0(l) = 0$. The next guess would be to write u' as

$$u' \approx u''(l)(x-l) + \dots,$$

but $u'' = w$, which we know varies, so $u''(l)$ is ill-defined. Hence there must be a layer in the *derivative* of u' . So we write

$$u' = \epsilon^a f(\xi).$$

But then in order to match we must have

$$\frac{\epsilon^a}{\epsilon^b} \frac{df}{d\xi}(-\infty) = \frac{d(u')}{dx}(l^-) = v(l^-) = O(1),$$

so $a = b$. But this essentially means that $f' = w$. So we have

$$\epsilon^{1-2b}w'' + fw' = 0, \quad w(-\infty) = v_0, \quad w(\infty) = \alpha^2v_0, \quad (\text{D})$$

and for a dominant balance we have $b = 1/2$. Hence this is the full equation on an infinite domain. Note that $\text{sgn}(f) = \text{sgn}(\xi)$, so as $\xi \rightarrow -\infty$, we have $w'' - |f(-\infty)|w' = 0$, which has the proper decay, and as $\xi \rightarrow \infty$, we have $w'' + |f(\infty)|w' = 0$, which also has the proper decay.

Can there be more than one internal layer? No, since if there were two internal layers (say at l_1 and l_2), then $u'(l_1) = u'(l_2) = 0$. But u is a quadratic, which has at most one extremum. Hence there is only one internal layer and the layer so derived is unique.

- (b) (4 points) Construct any necessary outer solutions and indicate the position of any layers.

Solution. With the internal layer at $x = l$, we see that we must patch together the following two outer solutions on either side of $x = l$:

$$\begin{aligned} v_L(x) = 2v_* & \implies u_L(x) = v_*x(x + A_L l), & 0 < x < l, \\ v_R(x) = 2\alpha^2 v_* & \implies u_R(x) = \alpha^2 v_*(x - 1)[(x - 1) + A_R(l - 1)], & l < x < 1. \end{aligned}$$

As noted above, $u'(l) = 0$, so we have

$$\begin{aligned} u'_L(l) = v_*l(2 + A_L) = 0 & \implies A_L = A_R = -2. \\ u'_R(l) = \alpha^2 v_*(l - 1)(2 + A_R) = 0 & \end{aligned} \quad (\text{E})$$

Moreover, u must be continuous at $x = l$, since the only layer is in v . Hence we have

$$u_L(l) = -v_*l^2 = -\alpha^2 v_*(l - 1)^2 = u_R(l),$$

where we have used (E). Continuing to simplify, we obtain

$$\begin{aligned} (1 - \alpha^2)l + 2\alpha^2 l - \alpha^2 &= 0 \\ [(1 + \alpha)l - \alpha][(1 - \alpha)l + \alpha] &= 0 \\ l &= \frac{\alpha}{\alpha + 1}, \end{aligned}$$

where we have chosen the root in $(0, 1)$. Therefore, our solutions are given by

$$\begin{aligned} u_L(x) &= v_*x \left(x - \frac{2\alpha}{\alpha + 1} \right), & 0 < x < l, \\ u_R(x) &= \alpha^2 v_*(x - 1) \left[(x - 1) + \frac{2}{\alpha + 1} \right], & l < x < 1. \end{aligned}$$

(c) (3 points) Sketch the solution.

Solution. On the next page there are graphs for $\epsilon = 0.01$, $v_* = 1$, and $\alpha = 2$.

3. Now we wish to examine the system (4.2) coupled to

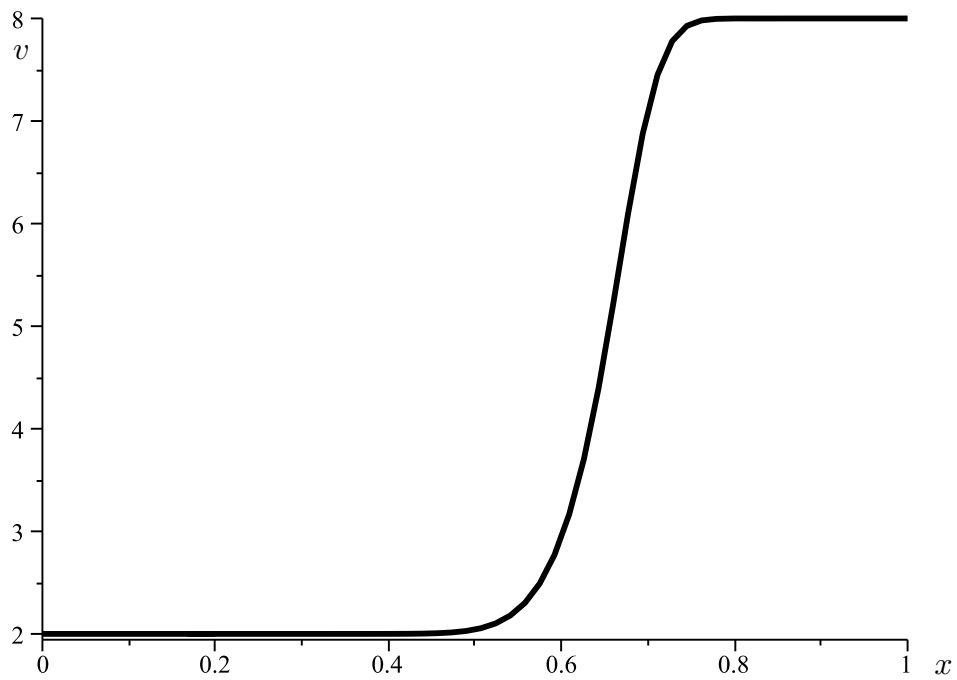
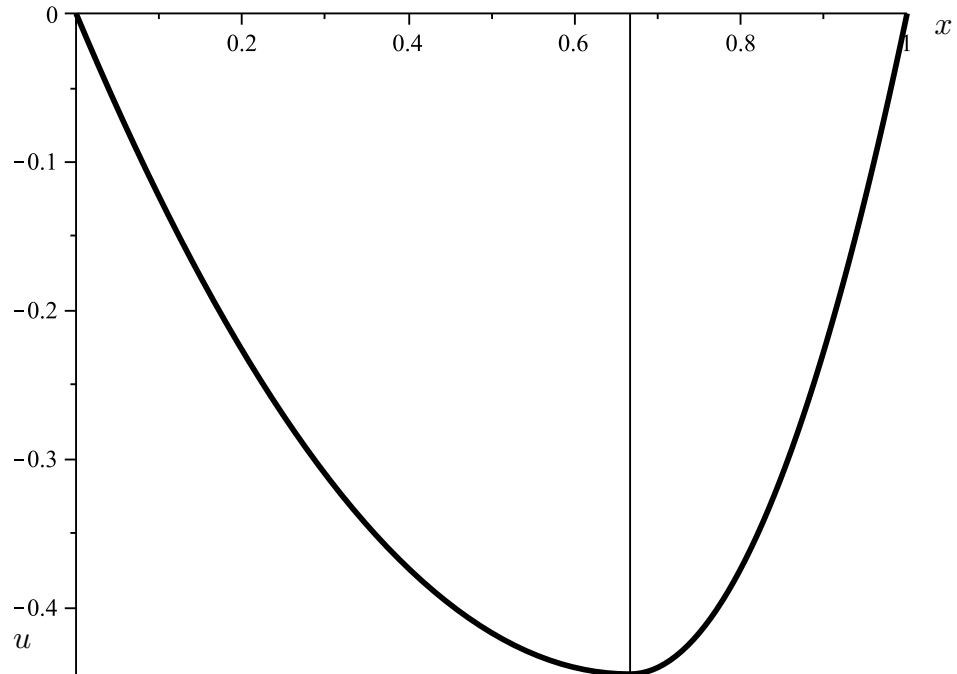
$$w'' = -v, \quad v(0) = 2(\bar{v} - v_*), \quad v(1) = 2(\bar{v} + v_*), \quad \bar{v} > v_* > 0. \quad (4.4)$$

which replaces (4.3).

(a) (4 points) Write down the proper scalings and identify the position for each needed layer. Explain why your choice of layers is unique.

Solution. We first examine if an internal layer can occur as in (D) above. In this case must let $u''(l) = -w(\xi)$ because of the change of sign in the first equation, so (D) becomes

$$w'' - fw' = 0, \quad w(-\infty) = v_{<} > 0, \quad w(\infty) = v_{>}, \quad v_{>} > v_{<},$$



where we have made the matching conditions more arbitrary. But now with the change of sign, we have $w'' + |f(-\infty)|w' = 0$, which does not have the proper decay. So no internal layers are possible.

Note these conditions imply that w and w' are positive. Now examine what happens as $\xi \rightarrow -\infty$. From the equation, we see that $w'' < 0$, so the graph is concave down, which violates the condition that w must asymptote to a constant as $\xi \rightarrow -\infty$. Hence no internal layers are possible.

Now looking at our analysis from before, we see that the analysis through (B) holds. Adapting (B) using the new minus sign in (4.4), we have

$$\begin{aligned} v_L(x) = 2(\bar{v} - v_*) &\implies u_L(x) = -(\bar{v} - v_*)x(x-1) &\implies u'_L(1) = -(\bar{v} - v_*) < 0 \\ v_R(x) = 2(\bar{v} + v_*) &\implies u_R(x) = -(\bar{v} + v_*)x(x-1) &\implies u'_R(0) = (\bar{v} + v_*) > 0 \end{aligned}$$

Equation (C.2), which is balanced with $b = 1$, remains

$$w'' + u'_0(l)w' = 0, \quad u'_0(l) \neq 0, \quad (\text{F})$$

but due to the sign change of u'_0 , we see that solutions of (D) will not blow up as we exit the layer. However, we see from part (a) that we cannot have a layer on just one side. Hence there must be a layer on both sides of width ϵ .

(b) (3 points) Explain why $v'(0) = v'(1)$.

Solution. Combining our two differential equations, we have

$$\begin{aligned} -\epsilon u^{(4)} - u' u^{(3)} &= 0 \\ \epsilon e^{u/\epsilon} \frac{du^{(3)}}{dx} + e^{u/\epsilon} u' u^{(3)} &= 0 \\ \frac{d}{dx} \left(\epsilon e^{u/\epsilon} u^{(3)} \right) &= 0 \\ \epsilon e^{u/\epsilon} v' &= \text{constant}. \end{aligned}$$

Since $u(0) = u(1)$, $v'(0) = v'(1)$.

(c) (5 points) Write down a leading-order uniformly valid solution for v and the leading-order outer solution for u .

Solution. Since there are no internal layers, there must be a single (constant) outer solution, as yet undetermined:

$$v_0(x) = 2V \quad \implies \quad u_0(x) = -Vx(x-1) \quad \implies \quad \begin{aligned} u'_0(1) &= -V \\ u'_0(0) &= V, \end{aligned}$$

and hence for the signs to work out, $V > 0$.

Letting

$$\xi_0 = \frac{x}{\epsilon}, \quad w_0(\xi_0) = v(x); \quad \xi_1 = \frac{x-1}{\epsilon}, \quad w_1(\xi_1) = v(x),$$

we obtain

$$\begin{aligned} w''_0 + Vw'_0 &= 0, & w_0(0) &= 2(\bar{v} - v_*), & w_0(\infty) &= 2V \\ w_0 &= 2V + 2(\bar{v} - v_* - V)e^{-V\xi_0}, \end{aligned} \quad (\text{G.1})$$

$$\begin{aligned} w''_1 - Vw'_1 &= 0, & w_1(0) &= 2(\bar{v} + v_*), & w_1(-\infty) &= 2V \\ w_1 &= 2V + 2(\bar{v} + v_* - V)e^{V\xi_1}. \end{aligned} \quad (\text{G.2})$$

As stated before, $v'(0) = v'(1)$, which implies that

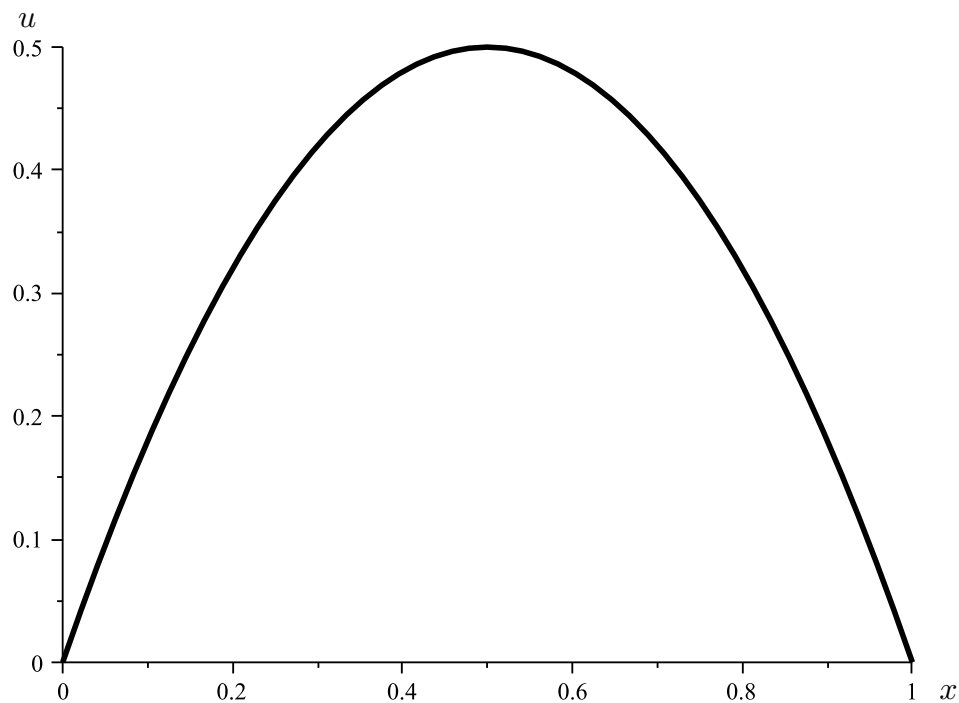
$$\begin{aligned} w'_0(0) &= w'_1(0) \\ -2V(\bar{v} - v_* - V) &= 2V(\bar{v} + v_* - V) \\ 2V &= 2\bar{v} \\ u_0(x) &= \frac{\bar{v}x(1-x)}{2}. \end{aligned} \tag{H}$$

Since $2V$ is the common part, we see that the uniform solution for V is given by

$$\begin{aligned} v(x) &= w_0(\xi_0) + w_1(\xi_1) - 2V \\ &= 2\bar{v} - 2v_* \exp\left(-\frac{\bar{v}x}{\epsilon}\right) + 2v_* \exp\left(\frac{\bar{v}(x-1)}{\epsilon}\right). \end{aligned}$$

(d) (3 points) Sketch the solution.

Solution. Here is the solution with $\bar{v} = 4$, $v_* = 2$, and $\epsilon = 0.01$.



M810S10Sol4.8

