Homework Set 3 Solutions

1. Consider the following equation:

$$\epsilon^4 y'' + x^2 y' + (x-\epsilon)y = 0, \qquad y(0) = y(1) = 1. \tag{3.1}$$

(a) (4 points) Construct any needed leading-order outer expansions.

Solution. Letting $y \sim y_0 + o(1)$, we have, to leading orders,

$$\begin{aligned} \epsilon^4 y_0'' + x^2 y_0' + (x-\epsilon)y_0 &= 0 \\ x y_0' + y_0 &= 0. \end{aligned} \qquad O(1)$$

Solving the $O(1)$ equation, we obtain

$$y_0(x) = \frac{A}{x},$$

and so we cannot satisfy the boundary condition at $x = 0$. Therefore, we satisfy the boundary condition at $x = 1$ to obtain

$$y_0(x) = \frac{1}{x}.$$

(b) (9 points) Construct any needed leading-order inner expansions.

Solution. Clearly we need a boundary layer at $x = 0$; therefore we let

$$z = \frac{x}{\epsilon^a}, \qquad y(x) = w(z).$$

Making these substitutions into (3.1), we obtain

$$\begin{aligned} \epsilon^{4-2a} w'' + \epsilon^a z^2 w' + (\epsilon^a z - \epsilon)w &= 0 \\ \epsilon^{4-2a} w'' + \epsilon^a (z^2 w' + zw) - \epsilon w &= 0. \end{aligned} \tag{A}$$

There are three possible balances. Balancing the first and second terms yields $a = 4/3$. But this is not a dominant balance, since those terms are $O(\epsilon^{4/3})$ while the last term is $O(\epsilon)$. Balancing the second and third terms yields $a = 1$. This is a dominant balance since those terms are $O(\epsilon)$ while the first term is $O(\epsilon^2)$. Thus we have

$$z^2 w' + zw - w + \epsilon w'' = 0.$$

To find the matching condition, we rewrite the outer solution in the inner variables:

$$y(x) \sim \frac{1}{\epsilon z}. \tag{B}$$

Thus we see that we should let $w = \epsilon^{-1} w_0$ in order to match. Since the equation is linear, scaling the dependent variable does not affect our previous calculations and we have

$$\begin{aligned} z^2 w_0' + (z-1) w_0 &= 0 \\ (z e^{1/z} w_0)' &= 0 \\ w_0(z) &= \frac{A e^{-1/z}}{z}. \end{aligned}$$

To match, we note that for large z, we have

$$w(z) \sim \frac{1}{\epsilon}\left[\frac{A}{z}\left(1 - \frac{1}{z}\right)\right],$$

and so to match with (B), $A = 1$. However, $w(0) = 0$, which does not satisfy the boundary condition. Thus there must be an additional layer corresponding to the third scaling. Balancing the first and third terms yields $a = 3/2$. This is a dominant balance since those terms are $O(\epsilon)$ while the first term is $O(\epsilon^{3/2})$. Thus we have

$$\begin{aligned} y(x) = u(\xi), \qquad \xi = \frac{x}{\epsilon^{3/2}}, \\ u'' - u = 0, \qquad u(0) = 1, \quad u(\infty) = 0, \end{aligned}$$

where the last condition comes from matching to the w solution. The solution of this equation is given by

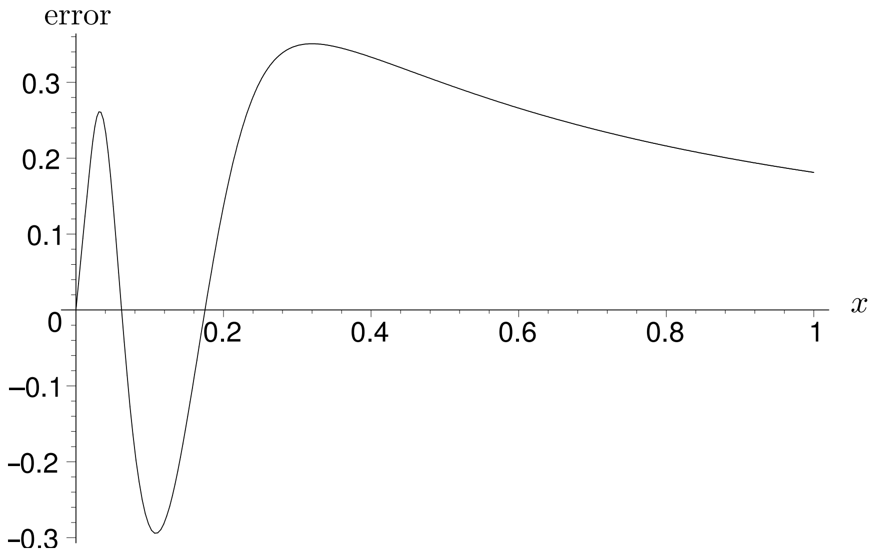
$$u(\xi) = e^{-\xi}.$$

(c) (2 points) Construct the leading order uniform expansion.

Solution. The common part between u and w is zero; when matching between between w and y we matched all of equation (B), which is all of the outer solution. Therefore the uniform solution is given just by adding u and w together:

$$\begin{aligned} y_{\text{u}} &= e^{-\xi} + \frac{e^{-1/z}}{\epsilon z} \\ &= \exp\left(-\frac{x}{\epsilon^{3/2}}\right) + \frac{1}{x}\exp\left(-\frac{\epsilon}{x}\right). \end{aligned}$$

(d) (5 points) Using a computer algebra system, plot the error between the numerical solution of (3.1) and your answer to (c) for $\epsilon = 0.2$. Discuss the size of the error in various regions in light of what the size of the next-order term would be in each region.



Solution. See above. The next term in the outer expansion should be $O(\epsilon)$. The next term in the intermediate expansion should be $O(1)$. The next term in the inner expansion should be $O(\epsilon^{1/2})$ (since the ϵ^a term in (A) is $O(\epsilon^{1/2})$ smaller than the dominant terms).

2. (3 points) Show that in the model nonlinear example presented in class:

$$\epsilon y'' + yy' - y = 0, \qquad y(0) = A, \qquad y(1) = B,$$

if we make the substitutions

$$B \mapsto -A, \qquad A \mapsto -B, \qquad x \mapsto 1 - x, \qquad y \mapsto -y,$$

we obtain the same differential equation as before.

Solution. Let

$$\bar{x} = 1 - x, \qquad \bar{y}(\bar{x}) = -y(x).$$

Then our equations become

$$-\epsilon \frac{d^2\bar{y}}{d\bar{x}^2} - \bar{y}\frac{d\bar{y}}{d\bar{x}} + \bar{y} = 0, \qquad \bar{y}(1) = -A, \qquad \bar{y}(0) = -B.$$

Now letting

$$\bar{B} = -A, \qquad \bar{A} = -B,$$

we have

$$\epsilon \frac{d^2\bar{y}}{d\bar{x}^2} + \bar{y}\frac{d\bar{y}}{d\bar{x}} - \bar{y} = 0, \qquad \bar{y}(0) = \bar{A}, \qquad \bar{y}(1) = \bar{B},$$

as required.

3. Consider the following problem:

$$\epsilon u'' - uu' - u = 0, \qquad 0 \le x \le 1, \qquad 0 < \epsilon \ll 1, \qquad (3.2a)$$

$$u(0) - \alpha\epsilon u'(0) = 2, \qquad u(1) = -2, \tag{3.2b}$$

where α is a positive $O(1)$ constant.

(a) (5 points) Construct any outer expansions to the order needed to satisfy the boundary conditions.

Solution. Letting $u \sim u_0 + \epsilon u_1 + \cdots$, we have

$$\begin{aligned} \epsilon(u_0 + \epsilon u_1)'' - (u_0 + \epsilon u_1)(u_0 + \epsilon u_1)' - (u_0 + \epsilon u_1) = 0 & \\ -u_0(u_0' + 1) = 0, & \qquad O(1) \\ u_0'' - u_1(u_0' + 1) - u_1' u_0 = 0, & \qquad O(\epsilon) \end{aligned}$$

$$\begin{aligned} (u_0 + \epsilon u_1)(0) - \alpha\epsilon(u_0 + \epsilon u_1)'(0) = 2 & \\ u_0(0) = 2, \quad u_0(1) = -2 & \qquad O(1) \\ u_1(0) = \alpha u_0'(0), \quad u_1(1) = 0, & \qquad O(\epsilon) \end{aligned}$$

Solving the first equation, we see that

$$u_0 \equiv 0 \text{ or } u_0 = A_0 - x.$$

Substituing the above result into the $O(\epsilon)$ equations, we have

$$\begin{aligned} u_1 \equiv 0 \text{ or } \; - u_1' u_0 &= 0 \\ u_1 &= A_1. \end{aligned}$$

Using these expressions, we obtain three possible cases, depending on which boundary conditions we solve. As before, we use the subscripts "L" and "R" to satisfy conditions which satisfy the left and right boundary conditions, respectively. We use "N" as a subscript for the solution that satisfies neither condition.

$$\begin{aligned} u_{\mathrm{L}}(x) &= 2 - x - \alpha\epsilon, \\ u_{\mathrm{R}}(x) &= -(1 + x), \\ u_{\mathrm{N}}(x) &\equiv 0. \end{aligned}$$

(b) (9 points) Describe completely any needed leading-order inner equations, and solve any whose solution is obvious.

Solution. This problem is exactly analogous to the example presented in class with the exception of the sign of the nonlinear term. Therefore, we see that the correct scaling is

$$\xi = \frac{x - x_{\mathrm{d}}}{\epsilon}, \qquad u(x) = w(\xi), \qquad \beta^2 = y^2(x_{\mathrm{L}}),$$

and the inner equation becomes

$$w'' - ww' = 0,$$

which has a solution given by the negative of the solutions presented in class. Therefore, we have

$$
\begin{aligned}
w_{\mathrm{t}} &= -\beta \tanh\left(\frac{\beta(\xi+k)}{2}\right), && |w| < |\beta|, \\
w_{\mathrm{c}} &= -\beta \coth\left(\frac{\beta(\xi+k)}{2}\right), && |w| > |\beta|, \\
w_0 &= -\frac{2}{\xi + C}, && \beta = 0.
\end{aligned}
$$

Note that in this case the solutions tend to $-\beta$ as $\xi \to \infty$.

Now we check the various layers to see what can occur. Suppose that u_{L} holds throughout the domain, and there is a boundary layer at $x = 1$ only. In that case, $\beta = 1$ and $w(0) = -2$, so we're in the coth case. But a coth layer won't work since $w(0)$ and β aren't of the same sign. Suppose that u_{R} holds throughout the domain, and there is a boundary layer at $x = 1$ only. In that case, $\beta = -1$ and

$$
\begin{aligned}
w(0) - \frac{\alpha\epsilon}{\epsilon} w'(0) &= 2 \\
w(0) - 2\alpha w'(0) &= 2. \qquad \text{(C)}
\end{aligned}
$$

A coth layer won't work since in such a layer $w(0) < 0$ and $w'(0) > 0$, so (C) won't be satisfied. Therefore, we check a tanh layer. Our solution is

$$
w_{\mathrm{t}} = -\tanh\left(\frac{\xi + k}{2}\right), \qquad \text{(D.1)}
$$

and satisfying (C) results in

$$
\begin{aligned}
-\tanh\left(\frac{k}{2}\right) + \frac{\alpha}{2}\operatorname{sech}^2\left(\frac{k}{2}\right) &= 2 \\
\frac{\sinh k}{2} + \cosh k &= \frac{\alpha}{2} - 1 \\
\frac{3e^k + e^{-k}}{4} &= \frac{\alpha}{2} - 1 \\
3K^2 + (4 - 2\alpha)K + 1 = 0, &\qquad K = e^k \\
K &= \frac{2\alpha - 4 \pm \sqrt{(4-2\alpha)^2 - 12}}{6}.
\end{aligned}
$$

For this equation to have two real solutions, we must have

$$
\begin{aligned}
4 - 2\alpha &< -2\sqrt{3} \\
\alpha &> 2 + \sqrt{3},
\end{aligned}
$$

where we have used the fact that $\alpha > 0$.

We next check the u_N solution. Trying to construct a boundary layer on the right, we have

$$w_0 = -\frac{2}{\xi+1},$$

which diverges for $\xi = -1$. Therefore, such a layer does not exist.

Lastly, we check to see if there can be an internal layer. At such a layer, $u_L(x_L) = -u_R(x_L)$, with $u_R(x_L) < 0$. This occurs when

$$2 - x_L = -(1 + x_L) \qquad x_L = \frac{1}{2}, \qquad \beta = \frac{3}{2}.$$

By centering the shock, we have the internal solution

$$w = -\frac{3}{2}\tanh\left(\frac{3\xi}{4}\right). \tag{D.2}$$

No other cases exist.

(c) (3 points) Construct and sketch the leading-order uniformly valid approximation of the solution(s).

Solution. For the solution (D.2) that always exists, the common part on the left is $3/2$, while the common part on the right is $-1/2$. Therefore, we may write our uniform solution as

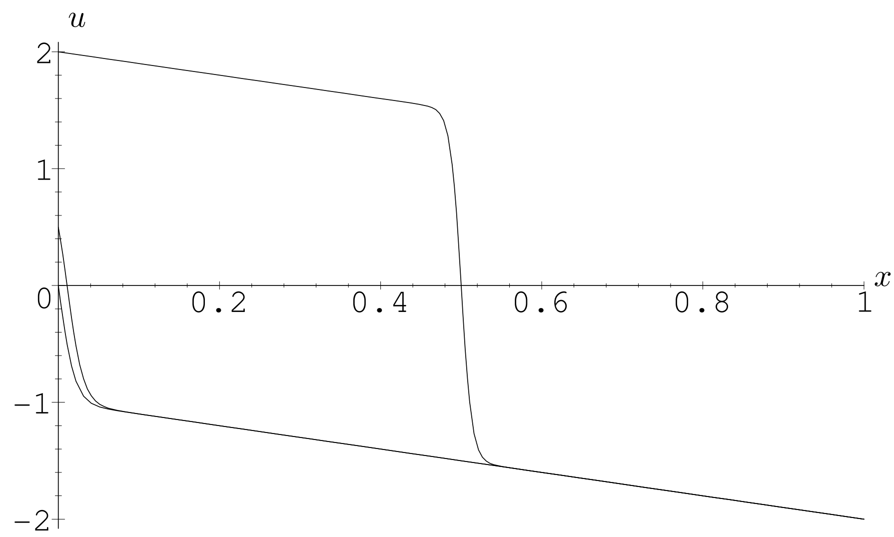
$$u_{u,I} = \frac{1}{2} - x - \frac{3}{2}\tanh\left(\frac{3(x-1/2)}{4\epsilon}\right) + O(\epsilon).$$

For the solutions that exist only for $\alpha > 2 + \sqrt{3}$, we have that the common part is -1, so we have

$$u_{u,II} = -x - \tanh\left(\frac{x + \epsilon k}{2\epsilon}\right),$$

$$\frac{\sinh(k/2)}{2} + \cosh(k/2) = \frac{\alpha}{2} - 1.$$

Here's a graph with $\epsilon = 0.01$ and $\alpha = 4$.



u
2
1
0
−1
−2
0.2
0.4
0.6
0.8
1
x