MATH 810-010
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Asymptotics and Perturbation Methods
Due: Mar. 1, 2010

Homework Set 2 Solutions

1. We wish to track a projectile's motion. In this case we neglect the changing gravitational force but include the effects of air resistance. The resulting dimensional equation is

$$m\frac{d^2\tilde{x}}{d\tilde{t}^2} + k\frac{d\tilde{x}}{d\tilde{t}} = -mg, \qquad \tilde{x}(0) = 0, \qquad \frac{d\tilde{x}}{d\tilde{t}}(0) = V. \tag{2.1}$$

   (a) (3 points) Using the appropriate scalings for a projectile near the Earth, obtain a dimensionless equation involving only the parameter

$$\beta \equiv \frac{kV}{mg}.$$

Solution. Since we are neglecting the changing gravitational force, we expect the time scale to be dictated by the constant gravitational acceleration and the initial velocity. The length scale would be this time scale times the velocity, so we have

$$\tilde{x} = \frac{V^2}{g}x, \qquad \tilde{t} = \frac{V}{g}t. \tag{A}$$

Substituting (A) into (2.1), we have

$$\begin{aligned} mg\frac{d^2x}{dt^2} + kV\frac{dx}{dt} &= -mg \\ \frac{d^2x}{dt^2} + \beta\frac{dx}{dt} &= -1, \qquad x(0) = 0, \qquad \frac{dx}{dt}(0) = 1. \end{aligned} \tag{B}$$

Since the effects of air resistance are quite weak, we let $\beta \ll 1$.

   (b) (7 points) Using this assumption, show that the dimensionless time and height corresponding to the highest position of the projectile is given by

$$(t_{\max}, x_{\max}) = \left(1 - \frac{\beta}{2}, \frac{1}{2} - \frac{\beta}{3}\right) + o(\beta).$$

Solution. Using the expansion $x \sim x_0 + \beta x_1 + \cdots$ in (B), we have

$$\frac{d^2}{dt^2}(x_0 + \beta x_1) + \beta\frac{d}{dt}(x_0 + \beta x_1) = -1$$

$$\frac{d^2x_0}{dt^2} = -1, \qquad O(1)$$

$$\frac{d^2x_1}{dt^2} + \frac{dx_0}{dt} = 0, \qquad O(\beta)$$

$$(x_0+\beta x_1)(0)=0, \qquad \frac{d}{dt}(x_0+\beta x_1)(0)=1$$

$$x_0(0)=0, \qquad \frac{dx_0}{dt}(0)=1, \qquad O(1)$$

$$x_1(0)=0, \qquad \frac{dx_1}{dt}(0)=0, \qquad O(\beta)$$

Solving these equations in order, we have

$$\frac{d^2x_0}{dt^2}=-1 \implies x_0=-\frac{t^2}{2}+At+B \implies x_0=-\frac{t^2}{2}+t,$$

$$\frac{d^2x_1}{dt^2}=t-1 \implies x_1=\frac{t^3}{6}-\frac{t^2}{2}+At+B \implies x_1=\frac{t^3}{6}-\frac{t^2}{2},$$

Thus, we have that

$$x=-\frac{t^2}{2}+t+\beta\left(\frac{t^3}{6}-\frac{t^2}{2}\right)+o(\beta). \qquad \text{(C)}$$

To solve for t_m, we recall that

$$\frac{dx}{dt}(t_m)=1-t_m+\beta\left(\frac{t_m^2}{2}-t_m\right)=0.$$

Letting $t_m=t_0+\beta t_1$, we have

$$1-(t_0+\beta t_1)+\beta\left[\frac{(t_0+\beta t_1)^2}{2}-(t_0+\beta t_1)\right]=0,$$

$$1-t_0=0, \qquad O(1)$$

$$-t_1+\frac{t_0^2}{2}-t_0=0, \qquad O(\beta)$$

Solving our equations, we have

$$t_0=1,$$

$$-t_1-\frac{1}{2}=0 \implies t_1=-\frac{1}{2}.$$

So

$$t_m=1-\frac{\beta}{2}+o(\beta),$$

as required. Obviously, there are other roots, but it is the one near 1 in which we are interested.

Substituting $t=t_m$ into (C) to obtain x_m, we have

$$\begin{aligned} x_m &= -\frac{1}{2}\left(1-\frac{\beta}{2}\right)^2+\left(1-\frac{\beta}{2}\right)+\beta\left[\frac{1}{6}\left(1-\frac{\beta}{2}\right)^3-\frac{1}{2}\left(1-\frac{\beta}{2}\right)^2\right] \\ &= -\frac{1}{2}(1-\beta)+1-\frac{\beta}{2}+\beta\left[\frac{1}{6}\left(1-\frac{3\beta}{2}\right)-\frac{1}{2}(1-\beta)\right] \\ &= \frac{1}{2}-\frac{\beta}{3}, \end{aligned}$$

as required.

2. (10 points) Find the roots of

$$\epsilon x^3 + x^2 - 1 = 0, \qquad 0 < \epsilon \ll 1$$

up to $O(\epsilon)$.

Solution. We begin by letting $x = x_0 + \epsilon x_1 + \cdots$. The assumption that the next order term in the expansion is $O(\epsilon)$ will be checked during the analysis. Then, to leading two orders we have

$$\begin{aligned} \epsilon(x_0 + \epsilon x_1)^3 + (x_0 + \epsilon x_1)^2 - 1 &= 0 \\ \epsilon x_0^3 + x_0^2 + 2\epsilon x_0 x_1 - 1 &= 0 \\ x_0^2 - 1 &= 0, && O(1) \\ x_0^3 + 2x_0 x_1 &= 0. && O(\epsilon) \end{aligned}$$

Solving our equations, we have

$$\begin{aligned} x_0^{\pm} = \pm 1 \qquad \Longrightarrow \qquad \pm 1 \pm 2x_1^{\pm} &= 0 \\ x_1^{\pm} &= -\frac{1}{2} \end{aligned}$$

$$x = \pm 1 - \frac{\epsilon}{2} + o(\epsilon).$$

Of course, we see that we have omitted one root. Letting $x = \epsilon^n y$, we have

$$\epsilon^{1+3n} y^3 + \epsilon^{2n} y^2 - 1 = 0,$$

from which we see that a dominant balance is $n = -1$. Now letting $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$, we have

$$\begin{aligned} \epsilon^{-2}(y_0 + \epsilon y_1 + \epsilon^2 y_2)^3 + \epsilon^{-2}(y_0 + \epsilon y_1 + \epsilon^2 y_2)^2 - 1 &= 0 \\ \epsilon^{-2}[y_0^3 + 3\epsilon y_1 y_0^2 + 3\epsilon^2(y_0^2 y_2 + y_0 y_1^2)] + \epsilon^{-2}[y_0^2 + 2\epsilon y_0 y_1 + \epsilon^2(2y_0 y_2 + y_1^2)] - 1 &= 0 \end{aligned}$$

$$\begin{aligned} y_0^3 + y_0^2 &= 0, && O(\epsilon^{-2}) \\ 3y_1 y_0^2 + 2y_0 y_1 &= 0, && O(\epsilon^{-1}) \\ 3(y_0^2 y_2 + y_0 y_1^2) + 2y_0 y_2 + y_1^2 - 1 &= 0. && O(1) \end{aligned}$$

Solving the first equation, we have $y_0 = 0, 0, -1$. Since we have tracked the two zero roots above, we focus on the root where $y_0 = -1$. Then we have

$$3y_1 - 2y_1 = 0 \qquad \Longrightarrow \qquad y_1 = 0$$

$$3y_2 - 2y_2 - 1 = 0 \qquad \Longrightarrow \qquad y_2 = 1$$

So the last root is

$$x = \epsilon^{-1}(-1+\epsilon^2) = -\epsilon^{-1} + \epsilon + o(\epsilon).$$

3. Consider the following problem, similar to one I presented in class:

$$\epsilon y'' + y' + y = 0, \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1,$$

$$y'(0) = A, \qquad y(1) = 1.$$

(a) (6 points) Calculate the outer expansion to $O(\epsilon)$.

Solution. We still expect the boundary layer to be near $x = 0$, and the other boundary condition is the same as it was as in the example presented in class. Hence, the outer solution is the same as well:

$$y(x) = e^{1-x}\left[1 + \epsilon(1-x)\right] + O(\epsilon).$$

(b) (8 points) Calculate the inner expansion to $O(\epsilon)$.

Solution. We let

$$\xi = \frac{x}{\epsilon}, \qquad y(x) = w_0(\xi) + \epsilon w_1(\xi) + \cdots.$$

The placement of our boundary layer does not affect the equations, and so we have the same equations as in problem 3:

$$\frac{d^2 w_0}{d\xi^2} + \frac{dw_0}{d\xi} = 0, \qquad O(\epsilon^{-1})$$

$$\frac{d^2 w_1}{d\xi^2} + \frac{dw_1}{d\xi} + w_0 = 0. \qquad O(1)$$

What has changed, however, is the boundary condition at $\xi = 0$:

$$\frac{1}{\epsilon}\frac{d}{d\xi}(w_0 + \epsilon w_1)(0) = A$$

$$\frac{dw_0}{d\xi}(0) = 0, \qquad \frac{dw_1}{d\xi}(0) = A.$$

Solving our $O(\epsilon^{-1})$ equation subject to our boundary condition, we have

$$w_0(\xi) = B + Ce^{-\xi} \qquad \Longrightarrow \qquad w_0(\xi) = B.$$

Using our matching condition, we see that $B = e$. Therefore, we see that we have the outer solution to $O(1)$. This is not surprising, since we have not imposed a condition which would have forced the function to change drastically near $x = 0$. However, we have imposed a condition which will make the *derivative* change drastically near 0, which should be reflected in the solution for w_1, since it is at this order at which that boundary condition appears.

Solving our $O(1)$ equation subject to our boundary condition, we have

$$w_1(\xi) = B + Ce^{-\xi} - e\xi \qquad \Longrightarrow \qquad w_1(\xi) = B - (A+e)e^{-\xi} - e\xi.$$

To determine B, we must expand both solutions in the same variables. We choose to express both in terms of the outer variables and then expand near $x = 0$. So we have

$$y = e^{1-x}\left[1 + \epsilon(1-x)\right] \sim e(1-x) + \epsilon e(1-x)^2$$

$$w = e + \epsilon\left[B - (A+e)e^{-x/\epsilon} - \frac{ex}{\epsilon}\right] \sim e + \epsilon B - ex$$

Examining the above, we see that we can match the terms

$$e - ex + \epsilon e \tag{D}$$

if we let $B = e$. So we have

$$w(\xi) = e + \epsilon\left[e - (A+e)e^{-x/\epsilon} - e\xi\right] + o(\epsilon).$$

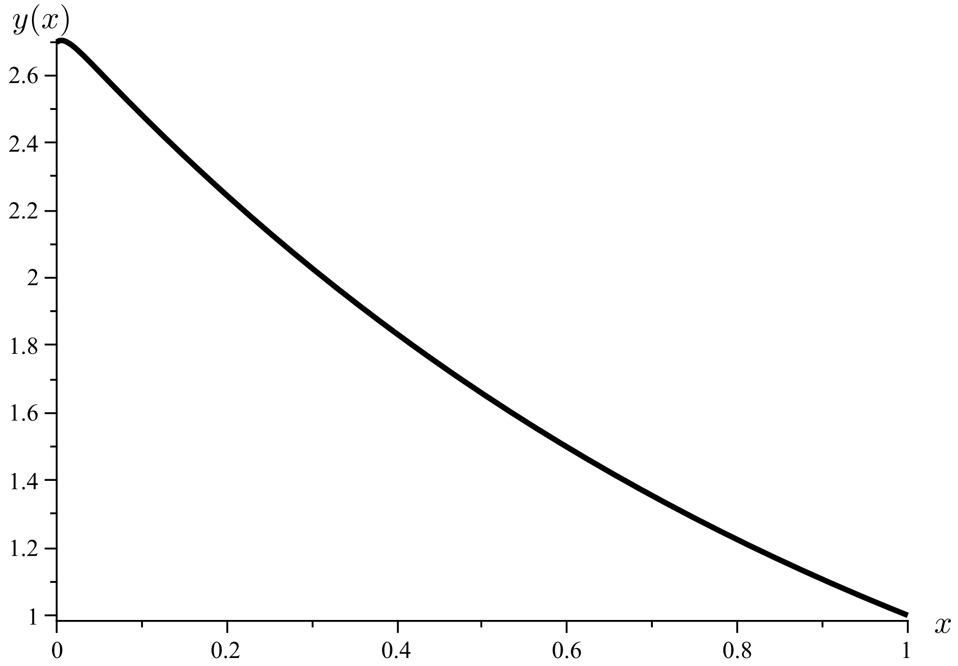
(c) (2 points) Calculate the uniform expansion to $O(\epsilon)$.

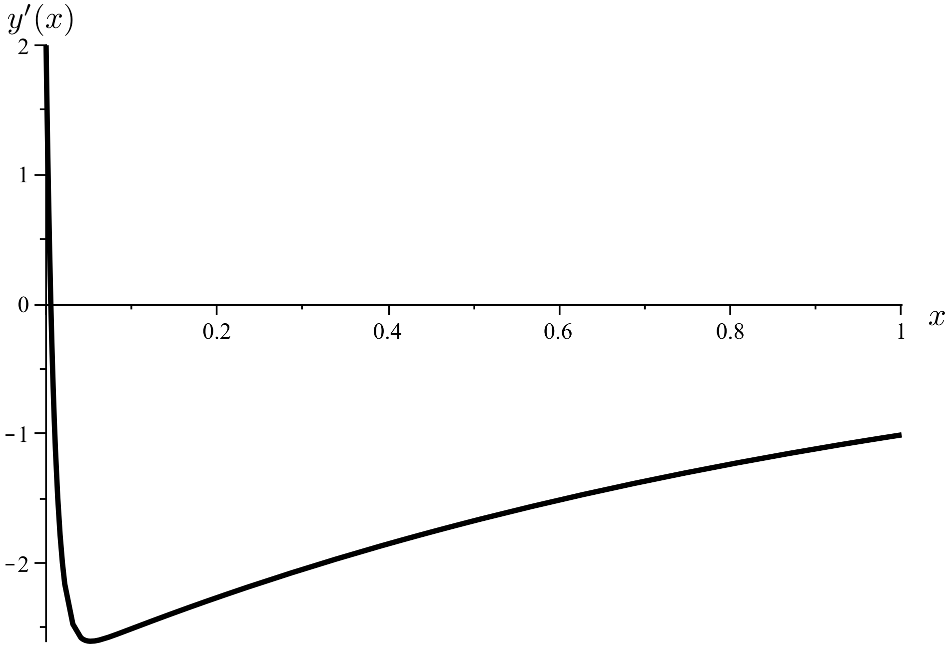
Solution. From our reasoning above, we see that the common part is given by (D). Therefore, we see that our uniform expansion is given by

$$\begin{aligned} y_u &= e^{1-x}\left[1 + \epsilon(1-x)\right] + e + \epsilon\left[e - (A+e)e^{-x/\epsilon} - \frac{ex}{\epsilon}\right] - (e - ex + \epsilon e) \\ &= e^{1-x}\left[1 + \epsilon(1-x)\right] - \epsilon(A+e)e^{-x/\epsilon} \end{aligned}$$

(d) (4 points) Use a computer algebra system to plot the *exact* solutions $y(x)$ and $y'(x)$ when $A = 2$ and $\epsilon = 0.01$.

Solution.




$y'(x)$
2
1
0
−1
−2
0.2
0.4
0.6
0.8
1
x