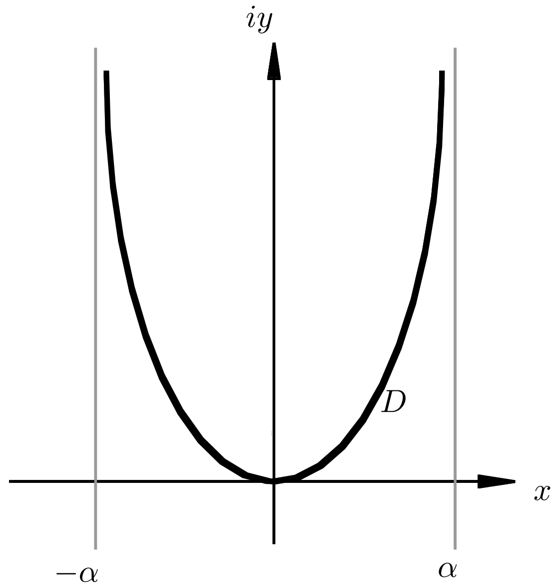
MATH 810-010 Asymptotics and Perturbation Methods
Prof. D. A. Edwards Due: May 17, 2010

Homework Set 11 Solutions



1. Let

$$I(k) = \int_D e^{ik\sin^3 z}\,dz, \qquad k \in \mathcal{R}, \tag{11.1}$$

where D is the contour shown in the figure above, and the contour is traversed from left to right.

(a) (3 points) Show that if $\pi/6 < \alpha < \pi/2$, the integral is guaranteed to exist.

Solution. The integral will converse when $\Re(i\sin^3 z) < 0$ as $|z| \to \infty$. Expanding $\sin x$ first for $z = x + iy$, we have

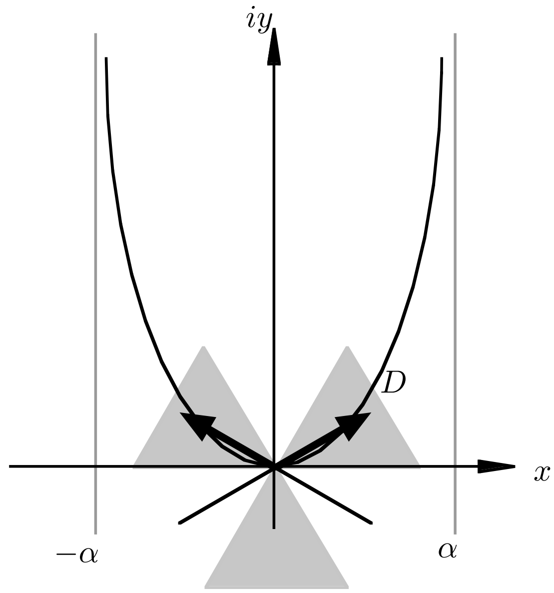
$$\sin z = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i\cos x \sinh y.$$

But

$$\sin^3 z = \frac{3\sin z - \sin 3z}{4}$$

by notes in class. Since $\sinh 3y \gg \sinh y$ as $y \to \infty$, we see that as $y \to \infty$,

$$\operatorname{sgn}(\Re(i\sin^3 z)) = -\operatorname{sgn}\Re(i\sin 3z)) = \operatorname{sgn}(\cos 3x \sinh y) = \operatorname{sgn}(\cos 3x), \qquad y \to \infty.$$



Since $\cos 3x < 0$ implies that $\pi/6 < |x| < \pi/2$, the result is proved.

(b) (8 points) Show that

$$I(k) \sim \frac{\Gamma(1/3)}{k^{1/3}\sqrt{3}}, \qquad k \to \infty.$$

Solution. Rewriting in standard saddle-point form, we have $\phi(z) = i \sin^3 z$, so $\phi'(z) = 3 \cos z \sin^2 z$, and the only saddle point of interest is $z = 0$. Letting $z = re^{i\theta}$ in (11.1) to find the steepest-descent paths, we have

$$I(k) = e^{i\theta} \int_D \exp\left(ikr^3 e^{3i\theta}\right)\, dr.$$

Thus on the steepest descent paths, $ie^{3i\theta} = -1$, so $\theta = \pi/6$, $\theta = 5\pi/6$, and $\theta = 3\pi/2$. If we deform onto the last ray, we can't return to the original contour (see figure). Thus we have

$$\begin{aligned} I(k) &\sim e^{5\pi i/6} \int_\infty^0 e^{-kr^3}\, dr + e^{\pi i/6} \int_0^\infty e^{-kr^3}\, dr = \left(e^{\pi i/6} - e^{5\pi i/6}\right) \int_0^\infty \frac{e^{-ku}}{3u^{2/3}}\, du \\ &= i\left(e^{-\pi i/3} - e^{\pi i/3}\right) \frac{\Gamma(1/3)}{3k^{1/3}} = \frac{2\Gamma(1/3)}{3k^{1/3}} \sin(\pi/3) = \frac{\Gamma(1/3)}{k^{1/3}\sqrt{3}}, \end{aligned}$$

as required.

2. (8 points) Show that

$$\sum_{k=0}^{n} \binom{n}{k} k! n^{-k} \sim \sqrt{\frac{\pi n}{2}} \text{ as } n \to \infty.$$

$$\textit{Hint: } k!n^{-k-1} = \int_0^\infty e^{-nx} x^k \, dx.$$

Solution. Using the hint, we obtain

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} k! n^{-k} &= \sum_{k=0}^{n} n \binom{n}{k} \int_0^\infty e^{-nx} x^k \, dx = n \int_0^\infty e^{-nx} \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} x^k \, dx \\ &= n \int_0^\infty e^{-nx} (1+x)^n \, dx = n \int_0^\infty e^{n[-x+\log(1+x)]} \, dx.\end{aligned}$$

Now we have a standard Laplace integral with

$$\phi(x) = -x + \log(1+x), \qquad \phi'(x) = -1 + \frac{1}{1+x}, \qquad \phi'(0) = 0, \qquad \phi''(0) = -1.$$

Therefore, using the formula given in class, we have

$$\sum_{k=0}^{n} \binom{n}{k} k! n^{-k} \sim n \frac{1}{2} \sqrt{\frac{2\pi}{n}} = \sqrt{\frac{n\pi}{2}}, \qquad n \to \infty,$$

where we have introduced the $1/2$ since $x = 0$ is an endpoint.

3. Consider the following equation and boundary conditions:

$$\epsilon u_{xx} = u_t + (\cos t) u_x, \qquad x > 0, \quad t > 0, \quad 0 < \epsilon \ll 1, \tag{11.2a}$$

$$u(x, 0) = 0, \qquad u(0, t) = 1. \tag{11.2b}$$

(a) (5 points) Construct the outer operator to the problem. Sketch the x-t plane, indicating any important subcharacteristics.

Solution. Letting $u \sim u_0 + o(1)$, we have

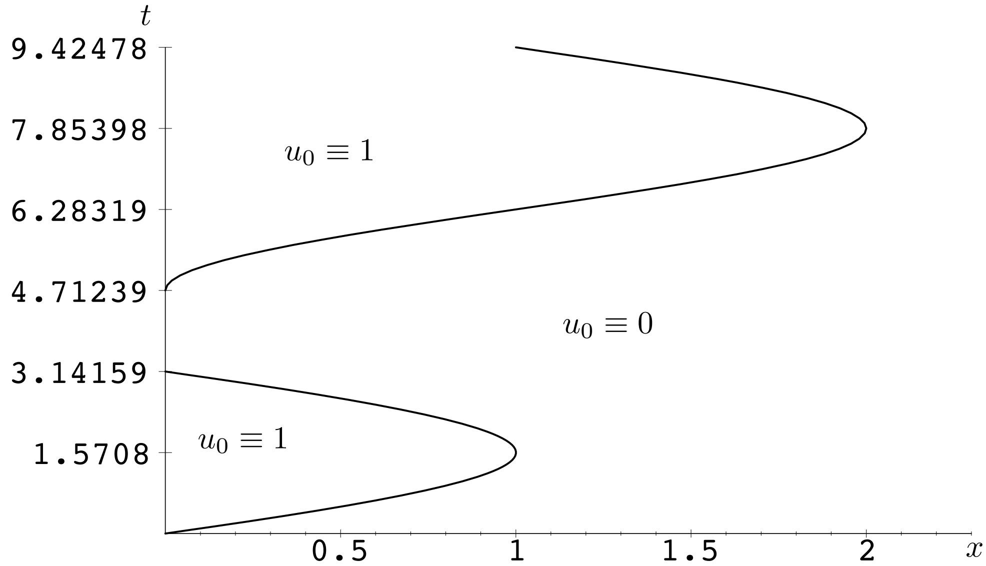
$$\begin{gathered}\frac{\partial u_0}{\partial t} + \cos t \frac{\partial u_0}{\partial x} = 0 \\ \frac{du_0}{dt} = 0 \text{ when } \frac{dx}{dt} = \cos t \\ u_0 = \text{constant when } x = \sin t + \xi.\end{gathered}$$

A sketch of the x-t plane is shown (it indicates the outer solution that shall be calculated in part (b)).

The important subcharacteristics are $x = \sin t$ for $t \in [0, \pi]$ and $x = \sin t + 1$ for $t \geq 3\pi/2$.

(b) (3 points) By considering the possibility of an initial layer, determine the outer solution.

Solution. There are several boundary layers that must be considered. However, we note that if there is an initial layer where the initial condition is smoothed to an outer



solution of $u_0 \equiv 1$, this would be the only layer necessary. (Such a layer is consistent with the infinite signal speed in a parabolic problem.) Therefore, we let

$$\tau = \frac{t}{\epsilon^\gamma}, \quad \gamma > 0; \qquad u(x,t) \sim f(x,\tau)$$

in (11.2a) to obtain

$$\epsilon f_{xx} = \epsilon^{-\gamma} f_\tau + f_x,$$

which means that there is no initial layer.

In order to construct the outer solution, it is easiest to divide the t domain into three parts: $t \in [0,\pi]$, $t \in [\pi, 3\pi/2]$, and $t > 3\pi/2$. Then from the diagram we have that

$$u_0 = \begin{cases} 1, & 0 \le x < \sin t, & 0 \le t \le \pi, \\ 0, & x > \sin t, & 0 \le t \le \pi, \\ 0, & & \pi \le t \le 3\pi/2, \\ 1, & 0 \le x < \sin t + 1, & t \ge 3\pi/2, \\ 0, & x > \sin t + 1, & t \ge 3\pi/2. \end{cases}$$

(c) (13 points) Construct any needed inner solutions. (Do not worry about corner layers.)

Solution. We see from the diagram that we need three boundary layers: one about $x = \sin t$ for $t \in [0,\pi]$, one about $x = 0$ for $t \in [0, 3\pi/2]$, and one about $x = \sin t + 1$ for $t \ge 3\pi/2$. We begin by letting

$$y = \frac{x - \sin t}{\epsilon^\alpha}, \quad \alpha > 0; \qquad u(x,t) \sim v(y,t) + o(1),$$

and note that this implies that

$$\frac{\partial}{\partial x} = \epsilon^{-\alpha}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \epsilon^{-\alpha}\cos t \frac{\partial}{\partial y}.$$

Making these substitutions into (11.2), we have

$$\epsilon^{1-2\alpha}\frac{\partial^2 v}{\partial y^2} = \frac{\partial v}{\partial t} - \epsilon^{-\alpha}\cos t\frac{\partial v}{\partial y} + \epsilon^{-\alpha}\cos t\frac{\partial v}{\partial y} \qquad \Longrightarrow \qquad \alpha = 1/2,$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial v}{\partial t}, \tag{A.1}$$

$$v(y,0) = H(-y), \qquad v(\infty,t) = 0, \qquad v(-\infty,t) = 1. \tag{A.2}$$

To solve equations (A) we use the standard similarity variable for the heat equation:

$$\zeta = \frac{y}{\sqrt{t}}, \qquad s(\zeta) = v(y,t).$$

Substituting this into (A), we have

$$s'' + \frac{\zeta}{2}s' = 0, \qquad s(\infty) = 0, \qquad s(-\infty) = 1,$$

the solution of which is

$$s(\zeta) = \frac{\operatorname{erfc}\zeta}{2} \qquad \Longrightarrow \qquad v(y,t) = \frac{1}{2}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{x-\sin t}{2\sqrt{\epsilon t}}\right).$$

For the region where $t > 3\pi/2$, we see that the boundary layer is periodic in t. However, the structure is the same as the previous layer. Therefore, we let

$$z = \frac{x-(\sin t + 1)}{\epsilon^{1/2}}, \qquad \tau_n = t - \frac{(4n-1)\pi}{2}, \quad n > 1; \qquad u(x,t) \sim v_n(z,\tau_n) + o(1)$$

in equations (11.2) to obtain

$$\frac{\partial^2 v_n}{\partial z^2} = \frac{\partial v_n}{\partial \tau_n}, \qquad v_n(y,0) = H(-y), \qquad v_n(\infty,\tau_n) = 0, \qquad v_n(-\infty,\tau_n) = 1.$$

Note that our choice of τ_n has forced the boundary conditions to remain the same for each layer. But of course this is exactly the same problem as in (A), so we have

$$v_n(z,\tau_n) = \frac{1}{2}\operatorname{erfc}\left(\frac{z}{2\sqrt{\tau_n}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{x-\sin t - 1}{2\sqrt{\epsilon[t-(4n-1)\pi/2]}}\right).$$

For the region where $t \in [\pi, 3\pi/2]$, we let

$$\xi = \frac{x}{\epsilon^\beta}, \quad \beta > 0; \qquad u(x,t) \sim w(\xi,t)$$

in (11.2) to obtain

$$\epsilon^{1-2\beta}\frac{\partial^2 w}{\partial \xi^2} = \frac{\partial w}{\partial t} + \epsilon^{-\beta}\cos t\frac{\partial w}{\partial \xi} \qquad \Longrightarrow \qquad \beta = 1,$$

$$\frac{\partial^2 w}{\partial \xi^2} = \cos t\frac{\partial w}{\partial \xi}, \tag{B.1}$$

$$w(0,t) = 1, \qquad w(\infty,t) = 0, \tag{B.2}$$

which is exactly the type of ODE boundary layer we expect. The solution is given by

$$w(\xi,t) = e^{\xi\cos t} = \exp\left(\frac{x\cos t}{\epsilon}\right),$$

where we recall that in this region, $\cos t < 0$.