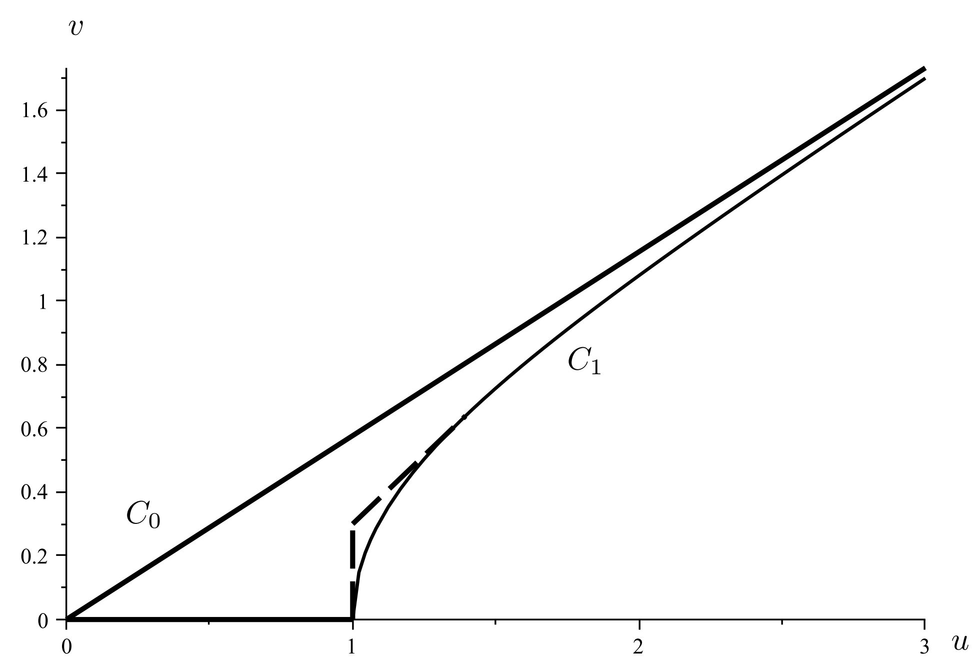
Homework Set 10 Solutions

1. (15 points) Use the method of steepest descents to find the full asymptotic behavior of
$$I(x) = \int_0^1 e^{ixt^3}\,dt = \int_0^1 e^{x\phi(t)}\,dt, \qquad x \to \infty.$$
You should find that
$$\int_0^1 e^{-xt^3}\,dt = (\sqrt{3}+i)\frac{\Gamma(1/3)}{6x^{1/3}} + e^{ix}\sum_{n=0}^{\infty}\frac{(2n)!}{n!(3ix)^{n+1}}.$$

Solution. We wish to transform our integral into two Watson's Lemma integrals with contours emanating from each of the endpoints. We begin by letting $t = u + iv$, so
$$\phi(t) = it^3 = i(u+iv)^3 = -3u^2v + v^3 + i(u^3 - 3uv^2).$$
Since $\phi(0) = 0$, contours of constant phase are given by $u^3 = 3uv^2$, which correspond to $u = 0$, $u = \pm v\sqrt{3}$. The latter two contours are closest to the original contour. Along both, $\phi(t) = -3v(3v^2) + v^3 = -8v^3$. Therefore, for a steepest descent contour we must have $v > 0$, so the correct contour is $C_0 : u = v\sqrt{3}$. (See diagram.) Thus $t = (\sqrt{3}+i)v$, and the integral in this region is
$$\int_{C_0} e^{x\phi(t)}\,dt = \int_0^\infty e^{-8xv^3}(\sqrt{3}+i)\,dv = (\sqrt{3}+i)\int_0^\infty e^{-xz}\frac{dz}{6z^{2/3}} = (\sqrt{3}+i)\frac{\Gamma(1/3)}{6x^{1/3}}.$$



Since $\phi(1) = i$, contours of constant phase are given by $u^3 - 3uv^2 = 1$, which we can solve for two curves $v(u)$ as solutions of a quadratic equation. By expanding for large u, we see that one of these curves approaches $u = v\sqrt{3}$ (*i.e.*, C_0) for large v. Because of the complicated nature of the full curve, we may rely on the discussion on page 285 to realize that we may deform the contour while making only a transcendentally small error. In particular, for small v, we take $u = 1 + \alpha v^2$ to obtain

$$\begin{aligned}(1+\alpha v^2)^3 - 3(1+\alpha v^2)^2 &= 1\\ 3\alpha v^2 - 3v^2 &= 0\\ \alpha &= 1\\ u &\sim 1 + v^2.\end{aligned}$$

Since for small v, $du/dv = 0$, we simplify even further, deforming the contour to the line $t = 1 + iv$ (see diagram). Then $dt = i\,dv$ and we have

$$\phi(t) = i(1+iv)^3 = -3v + v^3 + i(1-3v^2)$$
$$\int_{C_1} e^{x\phi(t)}\,dt = e^{ix}\int_0^\epsilon e^{-3xv(1-v^2)}e^{-3ixv^2}(i\,dv) + \text{TST},$$

where the transcendentally small terms come from the contour deformation. But in this region, $v^2 \ll 1$, so expanding the complex exponential in a Taylor series, we obtain

$$\begin{aligned}\int_{C_1} e^{x\phi(t)}\,dt &\sim e^{ix}\int_0^\epsilon e^{-3xv}\sum_{n=0}^\infty \frac{(-3ixv^2)^n}{n!}(i\,dv) \sim ie^{ix}\sum_{n=0}^\infty \frac{(-i)^n(3x)^n}{n!}\int_0^\infty e^{-3xv}v^{2n}\,dv\\ &\sim ie^{ix}\sum_{n=0}^\infty \frac{(3x)^n}{n!i^n}\frac{\Gamma(2n+1)}{(3x)^{2n+1}} = -e^{ix}\sum_{n=0}^\infty \frac{(2n)!}{n!i^{n+1}(3x)^{2n+1}}.\end{aligned}$$

Since there are no singularities to encounter when we deform the integral, we have

$$\begin{aligned}I + \int_{C_1} e^{x\phi(t)}\,dt &= \int_{C_0} e^{x\phi(t)}\,dt\\ I &= (\sqrt{3}+i)\frac{\Gamma(1/3)}{6x^{1/3}} + e^{ix}\sum_{n=0}^\infty \frac{(2n)!}{n!i^{n+1}(3x)^{2n+1}}\end{aligned}$$

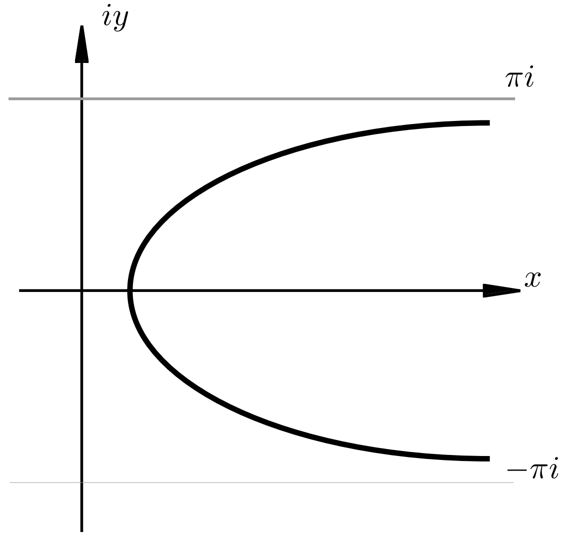
as required.

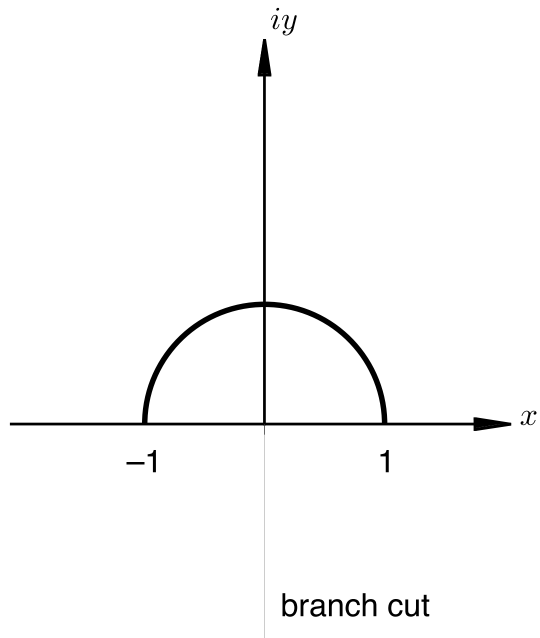
2. (10 points) The Bessel function of order ν may be represented as

$$J_\nu(x) = \frac{1}{2\pi i}\int_C e^{x\sinh t - \nu t}\,dt, \qquad (10.1)$$

where C is the contour shown in the left figure below, and the contour is traversed counterclockwise. Use this representation to show Debye's result

$$J_\nu(\nu\,\text{sech}\,\alpha) \sim \frac{e^{\nu(\tanh\alpha - \alpha)}}{\sqrt{2\pi\nu\tanh\alpha}} \text{ as } \nu\to\infty.$$





Solution. Letting $x = \nu \operatorname{sech} \alpha$ in (10.1), we obtain

$$J_\nu(\nu \operatorname{sech} \alpha) = \frac{1}{2\pi i} \int_C e^{\nu \phi(t)}\, dt, \qquad \phi(t) = \operatorname{sech} \alpha \sinh t - t,$$

$$\phi'(t) = \operatorname{sech} \alpha \cosh t - 1, \qquad c = \alpha, \qquad \phi''(c) = \tanh \alpha, \qquad \phi(c) = \tanh \alpha - \alpha,$$

where we have chosen the only saddle point in the domain of interest. Since $\arg(\phi''(c)) = 0$, we have the following result:

$$J_\nu(\nu \operatorname{sech} \alpha) \sim \frac{1}{2\pi i} \left[i e^{\nu(\tanh \alpha - \alpha)} \sqrt{\frac{2\pi}{\nu \tanh \alpha}} \right] = \frac{e^{\nu(\tanh \alpha - \alpha)}}{\sqrt{2\pi\nu \tanh \alpha}},$$

as required.

3. (15 points) Consider the function

$$I(z) = \int_D e^{z(s - \log s)}\, ds,$$

where D is the contour shown below (traversed counterclockwise) and we have chosen the branch where $\log 1 = 0$. Find the first term in the asymptotic expansion of $I(z)$ as $z \to \infty$ for all $0 \le \arg z < 2\pi$.

Solution. In terms of steepest descent contours, the complexity of z simply rotates them in the plane. Therefore, we work as if z is real and then interpret our results. In this case we have that

$$\phi(s) = s - \log s, \qquad \phi'(s) = 1 - s^{-1}, \qquad \phi''(s) = s^{-2},$$

$$c = 1, \qquad \phi(1) = 1, \qquad \phi''(1) = 1.$$

Using these results in the saddle point formula, we have

$$I(z) \sim \frac{i}{2}\sqrt{\frac{2\pi}{z}}e^z = ie^z\sqrt{\frac{\pi}{2z}},$$

where the extra 1/2 comes from the fact that the saddle point is also an endpoint. But now we note that for $\Re(z) < 0$, this contribution is very small. What's the contribution from the other endpoint? At this point, we use integration by parts to obtain

$$I(z) \sim \int_{-1}^{-1+\delta} \frac{1}{z(1-s^{-1})}\frac{d}{ds}\left(e^{z(s-\log s)}\right) ds \sim \left[\frac{e^{z(s-\log s)}}{z(1-s^{-1})}\right]_{-1}^{-1+\delta} \sim \frac{e^{-z(1+\pi i)}}{2z}.$$

Therefore, we see that when $\Re(z) < 0$, this contribution may be larger. To get the exact relationship, we write $z = x + yi$. Then if the contribution from the saddle point is larger, we must have that

$$\begin{aligned} \Re(x+yi) &\geq \Re(-(x+yi)(1+\pi i)) \\ x &\geq -x + \pi y \\ y &\leq \frac{2x}{\pi}. \end{aligned}$$

Therefore, we have that the dividing line between regions of validity is given by

$$\frac{y}{x} = \frac{2}{\pi},$$

or $\arg(z) = \alpha$, where $\alpha = \tan^{-1}(2/\pi)$. So we have

$$I(z) \sim \begin{cases} ie^z\sqrt{\dfrac{\pi}{2z}}, & \alpha - \pi \leq \arg(z) \leq \alpha, \\ \dfrac{e^{-z(1+\pi i)}}{2z}, & \alpha < \arg(z) < \alpha + \pi. \end{cases}$$

Since $I(z)$ is single-valued, we have covered all necessary arguments for z.