Homework Set 1 Solutions

1. (3 points) Verify that $x\sin x = O(x)$ as $x \to 0$ and as $x \to \infty$.

Solution.

$$\lim_{x\to 0} \frac{x\sin x}{x} = \lim_{x\to 0} \sin x = 0,$$

so $x\sin x = O(x)$ as $x \to 0$ (actually it is $o(x)$).

$$\lim_{x\to \infty} \frac{x\sin x}{x} = \lim_{x\to \infty} \sin x,$$

which does not exist. However, it varies in a bounded range. Using the more formal definition, we have that $x\sin x = O(x)$ because

$$|x\sin x| \le |x| \text{ for all } x,$$

and so in the definition $k(x) = 1$.

2. Verify the following equalities. α and β be are positive $O(1)$ constants and all calculations should be made in the limit that $\epsilon \to 0^+$.

(a) (2 points)

$$\epsilon = o\left(\epsilon|\log\epsilon|^{\beta}\right).$$

Solution.

$$\lim_{\epsilon\to 0^+} \frac{\epsilon}{\epsilon|\log\epsilon|^{\beta}} = \lim_{\epsilon\to 0^+} |\log\epsilon|^{-\beta} = 0.$$

(b) (2 points)

$$\epsilon|\log\epsilon|^{\beta} = o\left(\epsilon^{1-\alpha}\right).$$

Solution.

$$\lim_{\epsilon\to 0^+} \frac{\epsilon|\log\epsilon|^{\beta}}{\epsilon^{1-\alpha}} = \lim_{\epsilon\to 0^+} \frac{|\log\epsilon|^{\beta}}{\epsilon^{-\alpha}} = \lim_{\epsilon\to 0^+} \frac{\beta|\log\epsilon|^{\beta-1}}{-\alpha\epsilon^{-\alpha}}.$$

Repeated use of l'Hôpital's Rule shows that the numerator will eventually vanish, while the denominator grows without bound. Hence, the limit is 0, as required.

(c) (2 points)

$$e^{-\alpha/\epsilon} = o\left(\epsilon^{\beta}\right).$$

Solution. Let

$$\lim_{\epsilon\to 0^+} \frac{e^{-\alpha/\epsilon}}{\epsilon^{\beta}} = A.$$

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Then

$$\log A = \lim_{\epsilon\to 0^+} -\frac{\alpha}{\epsilon} - \beta\log\epsilon = \lim_{\epsilon\to 0^+} -\frac{\alpha}{\epsilon} \to -\infty,$$

where we have used (b). Therefore, we have that $A = 0$, as required.

3. Let $f(x)$ and $g(x)$ be functions at least as large as $O(1)$. (In other words, let $[f(x)]^{-1}$ and $[g(x)]^{-1}$ be $O(1)$.)

   (a) (5 points) Show by counterexample that $f(x) \sim g(x)$ does not necessarily imply that

$$e^{f(x)} \sim e^{g(x)}. \tag{1.1}$$

Solution. Let $f(x) = x^2 + x$, $g(x) = x^2$. Then, taking the behavior as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2+x}{x^2} = 1 \qquad \Longrightarrow \qquad f(x) \sim g(x).$$

But

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to\infty} \frac{e^{x^2+x}}{e^{x^2}} = \lim_{x\to\infty} e^{x} \to \infty \qquad \Longrightarrow \qquad e^{f(x)} \not\sim e^{g(x)}.$$

   (b) (3 points) Show that (1.1) does indeed hold if $f(x) = g(x) + o(1)$.

Solution. Let $f(x) = g(x) + o(1)$. Then, taking the behavior as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{g(x)+o(1)}{g(x)} = 1 \qquad \Longrightarrow \qquad f(x) \sim g(x).$$

Also,

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to\infty} \frac{e^{g(x)+o(1)}}{e^{g(x)}} = \lim_{x\to\infty} e^{o(1)} = 1 \qquad \Longrightarrow \qquad e^{f(x)} \sim e^{g(x)}.$$

4. (5 points) Find the first three nonzero terms in the expansion of the solution of

$$x^2 - 2.0004x + 0.9998 = 0$$

   using a perturbation approach. Compare your solution to that obtained from the exact root. Why does the straightforward analysis fail?

Solution. Let $\epsilon = 0.0002$. Then our equation becomes

$$x^2 - 2(1+\epsilon)x + (1-\epsilon) = 0. \tag{A}$$

As a first guess, we try the following form:

$$x \sim \sum_{n=0}^{\infty} x_n \epsilon^n.$$

Upon substitution into (A), we obtain, to leading orders,

$$\begin{aligned}(x_0+\epsilon x_1)^2-2(1+\epsilon)(x_0+\epsilon x_1)+(1-\epsilon)&=0\\(x_0^2+2\epsilon x_1)-2(1+\epsilon)x_0-2\epsilon x_1+(1-\epsilon)&=0\\(x_0-1)^2-\epsilon(2x_0+1)&=0\end{aligned}$$

Because of the cancellation of the two x_1 terms, the first two orders of the equation yield two inconsistent equations for x_0. This cancellation is a function of the equation, rather than the series. The most straightforward way to demonstrate this is, motivated by the next exercise, to let $z = x - 1$ in (A), which yields

$$\begin{aligned}(z+1)^2-2(1+\epsilon)(z+1)+(1-\epsilon)&=0\\z^2+2z+1-2(z+1)+1-\epsilon(2z+2+1)&=0\\z^2-\epsilon(2z+3)&=0.\end{aligned}\tag{B}$$

Since $z^2 = O(\epsilon)$, we try the following series:

$$z \sim \sum_{n=1}^{\infty} z_n \epsilon^{n/2}.$$

Substituting this series into (B), we obtain, to leading orders,

$$\begin{aligned}(\epsilon^{1/2}z_1+\epsilon z_2)^2-\epsilon[2(\epsilon^{1/2}z_1+\epsilon z_2)+3]&=0\\\epsilon[z_1^2+2\epsilon^{1/2}z_1z_2-3-2\epsilon^{1/2}z_1]&=0\\z_1^2&=3\qquad\Longrightarrow\qquad z_{1\pm}=\pm 3\\z_1z_2&=z_1\qquad\Longrightarrow\qquad z_2=1\\z_\pm&=\pm\epsilon^{1/2}\sqrt{3}+\epsilon\\x_\pm&=1\pm\epsilon^{1/2}\sqrt{3}+\epsilon.\end{aligned}$$

Now substituting our value of ϵ into the above, we have

$$x_+ = 1.024694897, \qquad x_- = 0.9757051026,$$

while the "exact" answers from Maple are given by

$$x_+ = 1.024695714, \qquad x_- = 0.9757042861.$$

If we didn't see the z substitution, we can try the following series:

$$x \sim \sum_{n=0}^{\infty} x_n \epsilon^{\alpha n}.$$

If we substitute this series into (A), we obtain, to leading orders,

$$(x_0 + \epsilon^\alpha x_1 + \epsilon^{2\alpha} x_2 + \epsilon^{3\alpha} x_3)^2 - 2(1+\epsilon)(x_0 + \epsilon^\alpha x_1 + \epsilon^{2\alpha} x_2 + \epsilon^{3\alpha} x_3) + (1-\epsilon) = 0$$

$$\begin{aligned}
&[x_0^2 + 2\epsilon^\alpha x_1 + \epsilon^{2\alpha}(x_1^2 + 2x_0x_2) + \epsilon^{3\alpha}(2x_1x_2 + 2x_0x_3)] \\
&\qquad - 2[x_0 + \epsilon^\alpha x_1 + \epsilon^{2\alpha} x_2 + \epsilon^{3\alpha} x_3 + \epsilon(x_0 + \epsilon^\alpha x_1 + \epsilon^{2\alpha} x_2 + \epsilon^{3\alpha} x_3)] + (1-\epsilon) = 0 \\
&(x_0 - 1)^2 + \epsilon^{2\alpha}(x_1^2 + 2x_0x_2 - 2x_2) + \epsilon^{3\alpha}(2x_1x_2 + 2x_0x_3 - 2x_3) - \epsilon(2x_0 + 1) - 2\epsilon^{1+\alpha} x_1 = 0.
\end{aligned}$$

Therefore, from the leading order we have that $x_0 = 1$ and to obtain a balance at next order we must have $\alpha = 1/2$. Making these substitutions, we have

$$\begin{aligned}
\epsilon[x_1^2 + 2x_2 - 2x_2 - (2+1)] + \epsilon^{3/2}(2x_1x_2 + 2x_3 - 2x_3 - 2x_1) &= 0 \\
x_1^2 &= 3 \\
x_2 &= 1,
\end{aligned}$$

as obtained above.

5. (7 points) Formulate a perturbation procedure to solve the equation $(x+1)^n = \epsilon x$. How rapidly do the roots vary with ϵ? Why?

Solution. Motivated by our work that led to (B), we let $z = x + 1$ to obtain

$$z^n = \epsilon(z-1).$$

Clearly the leading order term for z is $O(\epsilon^{1/n})$. Therefore, the roots vary like $x = -1 + O(\epsilon^{1/n})$.

6. Consider the function

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt.$$

(a) (9 points) Using repeated integration by parts, show that

$$\operatorname{erfc} z \sim \frac{e^{-z^2}}{z\sqrt{\pi}} \left[1 + \sum_{n=1}^\infty (-1)^n \frac{(1)(3)\cdots(2n-1)}{(2z^2)^n}\right], \qquad z \to \infty. \tag{1.2}$$

Solution. We begin by defining the integral $I_n(z)$:

$$I_n(z) = \int_z^\infty \frac{(-1)^{n+1}}{2^n t^{2n+1}} \frac{d}{dt}(e^{-t^2})\, dt.$$

Hence, we see that $\operatorname{erfc} z = I_0(z)/\sqrt{\pi}$. Integrating by parts, we have

$$\begin{aligned}
I_n(z) &= \left[\frac{(-1)^{n+1} e^{-t^2}}{2^n t^{2n+1}}\right]_z^\infty - [-(2n+1)] \int_z^\infty \frac{(-1)^{n+1}}{2^n t^{2n+2}} e^{-t^2}\, dt \\
&= \frac{(-1)^n e^{-z^2}}{z(2z^2)^n} + (2n+1) \int_z^\infty \frac{(-1)^{n+2}}{2^{n+1} t^{2n+3}} \frac{d}{dt}(e^{-t^2})\, dt \\
&= \frac{(-1)^n e^{-z^2}}{z(2z^2)^n} + (2n+1) I_{n+1}(z).
\end{aligned}$$

Using this recursion relation, we have

$$I_n(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \left[1 + \left(-\frac{1}{2z^2} + 3 \left(\frac{1}{2z^2} + 5 \left(-\frac{1}{2z^2} + \cdots \right] \right. \right. \right. ,$$

and hence we obtain (1.2).

To prove that this series is asymptotic, it suffices to show that $I_{n+1} = o(I_n)$ as $z \to \infty$. We note that for large z, we have

$$|I_{n+1}(z)| = \int_z^\infty \frac{1}{2^{n+1}t^{2n+3}} \frac{d}{dt}(e^{-t^2})\, dt \le \frac{1}{2z^2} \int_z^\infty \frac{1}{2^n t^{2n+1}} \frac{d}{dt}(e^{-t^2})\, dt = \frac{|I_n(z)|}{2z^2},$$

and hence the result is proven.

(b) (2 points) Using the ratio test outlined in class, estimate the optimal number of terms in the expansion for $z = 2$, $z = 3$, and $z = 4$.

Solution. From examination of (1.2), one sees that

$$|a_{n+1}| = \frac{2n+1}{2z^2} |a_n|,$$

where a_n is the nth term in the expansion. Examining where this ratio is greater than 1, we see that the estimate for the optimal value n is the first one for which

$$2n_*(z) + 1 > 2z^2,$$

where $n_*(z)$ is our estimate for the optimal value of n for a given value of z. Hence, we have that

$$n_*(2) = 4, \qquad n_*(3) = 9, \qquad n_*(4) = 16.$$

Since we're starting from $n = 0$, the actual number of terms is one more.