Homework Set 5

Read sections 11.1–11.3. You may find the following identities useful:

$$\sin a \cos b = \frac{\sin(a+b)+\sin(a-b)}{2}, \qquad \sin a \sin b = \frac{\cos(a-b)-\cos(a+b)}{2},$$

$$\cos a \cos b = \frac{\cos(a-b)+\cos(a+b)}{2}.$$

Two-Timing

1. Consider the following problem:

$$\ddot{u} + u + \epsilon u^2 = 0, \qquad 0 < \epsilon \ll 1, \qquad u(0) = \alpha, \qquad \dot{u}(0) = 0.$$

   (a) (4 points) At what order of ϵ do secular-causing terms first appear? You need not work through all the details.
   (b) (11 points) Show that the $O(1)$ uniformly valid approximation for this problem is

$$F_0(T,\tau) = \alpha \cos\left(T - \frac{5\alpha^2\tau}{12}\right)$$

   for appropriately chosen T and τ.

   Note that since this is a nonlinear problem, the phase shift depends on the amplitude.

(Continued on reverse.)

Question #2 should be treated as a "take-home exam." You may not ask anyone for help with the question, or use any aids besides your notes and book.

2. (25 points) For the following problem:

$$\ddot{u} + u - \epsilon u^2 = \epsilon \cos t, \qquad 0 < \epsilon \ll 1,$$

calculate the following:

(a) the proper expansion for $u(t;\epsilon)$,
(b) the proper slow-time scale τ, and
(c) the proper evolution equations for $A_0(\tau)$ and $B_0(\tau)$.

Remarks.
(i) For part (c), you should find that

$$A_0(\tau) \equiv -\left(\frac{6}{5}\right)^{1/3}, \qquad B_0(\tau) \equiv 0,$$

are steady-states of the evolution equations.
(ii) Note the strong similarity between the operators in this problem and problem 1. Use that fact and your solution to that problem to assist you.
(iii) This will probably be the most difficult homework problem I will assign you. For maximal credit, indicate the ways you attacked the problem, and how each way proved to be incorrect.