Supplemental Study Material Solutions

1. Find a Jordan basis and Jordan matrix for the following transformations:
   (a) $V = \mathcal{R}^3$, $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 + 3x_2, -x_1 + x_2 + 2x_3)$.

Solution. Solving for the eigenvalues, we rewrite in the standard basis:

$$A = {}_{E\leftarrow}T_E = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

$$p_T(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)[(1-\lambda)(3-\lambda)+1] = (2-\lambda)(4-4\lambda+\lambda^2)$$
$$= (2-\lambda)^3,$$

so $\lambda = 2$ is the only eigenvalue. Solving for the eigenvectors, we find two:

$$(A-2I)\mathbf{z}_1 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \mathbf{z}_1 = \mathbf{0}$$
$$\mathbf{z}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We note that the third component of $(A-2I)\mathbf{v}$ is 0 for any $\mathbf{v}$. Thus there is no solution to $(A-2I)\mathbf{v} = \mathbf{e}_3$, so the generalized eigenvector must be associated with $(1, 1, 0)$. Solving this equation, we have

$$(A-2I)\mathbf{z}_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \mathbf{z}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$\mathbf{z}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since the generalized eigenvector goes second in the block, we have that a Jordan basis is given by $W = \{(1, 1, 0), \mathbf{e}_2, \mathbf{e}_3\}$, and

$${}_{W\leftarrow}T_W = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(b)

$$V = \Big\{ \text{ solutions of } y^{(4)} + 7y^{(3)} + 18y'' + 20y' + 8y = 0 \Big\}, \quad T(y) = y'$$

Solution. By the definition of the operator and the space, the characteristic polynomial is given by

$$\begin{aligned} p_T(\lambda) &= \lambda^4 + 7\lambda^3 + 18\lambda^2 + 20\lambda + 8 = 0 = (\lambda+2)(\lambda^3 + 5\lambda^2 + 8\lambda + 4) \\ &= (\lambda+2)^2(\lambda^2 + 3\lambda + 2) = (\lambda+2)^3(\lambda+1), \end{aligned}$$

so our eigenvalues are -2 and 1. Thus a basis for V is given by $\{t^2e^{-2t}, te^{-2t}, e^{-2t}, e^{-t}\}$. We suspect from the theory of ODEs that t^2e^{-2t} is a generalized eigenvector of order 2; to verify it we compute

$$\begin{aligned} (T+2I)(t^2e^{-2t}) &= (2t - 2t^2 + 2t^2)e^{-2t} = 2te^{-2t}, \\ (T+2I)^2(t^2e^{-2t}) &= (T+2I)(2te^{-2t}) = (2 - 4t + 4t)e^{-2t} = 2e^{-2t}. \end{aligned}$$

But then by theorem 8.47, we have computed a Jordan basis for $\lambda = -2$. Thus we have that the Jordan basis is $W = \{2e^{-2t}, 2te^{-2t}, t^2e^{-2t}, e^{-t}\}$, and the Jordan matrix is

$$_W T_W = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Let $T \in \mathcal{L}(V)$, A the Jordan matrix for T, Λ the diagonal matrix such that $\lambda_{ii} = a_{ii}$ for all i, $M = A - \Lambda$. Prove that

(a) $M^m = O$ for some $m, M^k \neq O$ for $k < m$.

Solution. M is an upper triangular matrix with zeroes along the diagonal. Thus the eigenvalues of M are all equal to zero. Thus $p_M(\lambda) = \lambda^n$, $q_M(M) = M^m$ for some m.

(b) $M\Lambda = \Lambda M$

Solution. The ith row of M can always be written as $\lambda_{ii}\mathbf{e}_i + a\mathbf{e}_{i+1}$, where a is either 0 or 1. The jth column Λ is given by $\lambda_{jj}\mathbf{e}_j$. Thus

$$(M\Lambda)_{ij} = \langle \lambda_{ii}\mathbf{e}_i + a\mathbf{e}_{i+1}, \lambda_{jj}\mathbf{e}_j \rangle = \lambda_{ii}^2\delta_{ij} + a\lambda_{jj}\delta_{i+1,j}$$

The ith row Λ is given by $\lambda_{ii}\mathbf{e}_i$. The jth column of M can always be written as $\lambda_{jj}\mathbf{e}_j + b\mathbf{e}_{j-1}$, where b is either 0 or 1. Thus

$$\begin{aligned} (\Lambda M)_{ij} &= \langle \lambda_{ii}\mathbf{e}_i, \lambda_{jj}\mathbf{e}_j + b\mathbf{e}_j \rangle = \lambda_{ii}^2\delta_{ij} + b\lambda_{jj}\delta_{i,j-1} \\ (M\Lambda - \Lambda M)_{ij} &= (\lambda_{ii}^2\delta_{ij} + a\lambda_{jj}\delta_{i+1,j}) - (\lambda_{ii}^2\delta_{ij} + b\lambda_{ii}\delta_{i,j-1}) \\ &= (a\lambda_{i+1,i+1} - b\lambda_{ii})\delta_{i+1,j}, \end{aligned} \tag{A}$$

where we have used the fact that $\delta_{i,j-1} = \delta_{i+1,j}$. In order for a and b to be nonzero, the element must lie in a Jordan block. Thus $a = 1$ only if $\lambda_{ii} = \lambda_{i+1,i+1}$. Similarly, $b = 1$ only if $\lambda_{j-1,j-1} = \lambda_{jj}$. Since $j = i + 1$ for (A) to be nonzero, we have

$$(M\Lambda - \Lambda M)_{ij} = (a - b)\lambda_{ii}\delta_{i+1,j} = 0$$

for all i and j.

(c)

$$A^j = \sum_{i=0}^{r} \binom{j}{i} M^i \Lambda^i, \quad r = \max\{j, m-1\}.$$

Solution. By the Binomial Theorem,

$$A^j = (M + \Lambda)^j = \sum_{i=0}^{j} \binom{j}{i} M^i \Lambda^i,$$

where where we have used the fact that M and Λ commute. But $M^m = O$ by part (a) for some least integer m, so we may truncate the sum at $m - 1$ if $j \geq m$.

3. Find all values of α such that the transformation $T \in \mathcal{L}(\mathcal{R}^2), T(x_1, x_2) = (3x_1 + 4x_2, 4x_1 + \alpha x_2)$ is positive.

Solution. Constructing the inner product, we have

$$\begin{aligned}\langle T(x_1, x_2), (x_1, x_2)\rangle &= x_1(3x_1 + 4x_2) + x_2(4x_1 + \alpha x_2) = 3x_1^2 + 8x_1x_2 + \alpha x_2^2 \\ &= 3\left(x_1 + \frac{4x_2}{3}\right)^2 + \left(\alpha - \frac{16}{3}\right) x_2^2.\end{aligned}$$

Thus T is positive $\alpha \geq 16/3$.

4. Let $T, U \in \mathcal{L}(V)$ be positive semidefinite operators. Find conditions (if any) under which the following statements are true:

(a) If $T^2 = U^2$, $T = U$.

Solution. Let $Z = \{\mathbf{z}_j\}_1^n$ be an orthonormal basis of V consisting of eigenvectors of T. Then if

$$\mathbf{v} = \sum_{j=1}^{n} c_j \mathbf{z}_j,$$

we obtain

$$\begin{aligned}T\mathbf{v} &= \sum_{j=1}^{n} c_j T\mathbf{z}_j = \sum_{j=1}^{n} c_j \lambda_j \mathbf{z}_j, \quad \lambda_j \geq 0, \\ T^2\mathbf{v} &= \sum_{j=1}^{n} c_j \lambda_j^2 \mathbf{z}_j = U^2\mathbf{v},\end{aligned}$$

which implies that Z must also consist of eigenvectors of U^2. Thus

$$(U^2 - \lambda_j^2 I)\mathbf{z}_j = (U + \lambda_j I)(U - \lambda_j I)\mathbf{z}_j = \mathbf{0}.$$

Thus $\mathbf{z}_j$ is an eigenvector corresponding to $\pm\lambda_j$. But U is positive semidefinite, So $\mathbf{z}_j$ must be an eigenvector corresponding to λ_j, so

$$U\mathbf{v} = \sum_{j=1}^{n} c_j \lambda_j \mathbf{z}_j, \qquad U = T.$$

(b) TU is a positive semidefinite operator.

Solution. To be positive semidefinite, TU must be self-adjoint, so

$$TU = (TU)^* = U^* T^* = UT,$$

where we have used the fact that T and U are self-adjoint. Thus T and U must commute. Using Z as in part (a), we have

$$TU\mathbf{z}_j = UT\mathbf{z}_j = U(\lambda_j \mathbf{z}_j) = \lambda_j U\mathbf{z}_j,$$

so $U\mathbf{z}_j$ is an eigenvector for T corresponding to λ_j. Thus $U\mathbf{z}_j \in \operatorname{Span} Z_j$, where $T\mathbf{z} = \lambda_j \mathbf{z}$ for all $\mathbf{z} \in Z_j$. (Note that dim Z_j can be greater than 1 if the λ_j are not distinct.) U restricted to Z_j is an operator, and this operator can be diagonalized because U can. Therefore without loss of generality we can let $U\mathbf{z}_j = \alpha_j \mathbf{z}_j$. Then

$$TU\mathbf{z}_j = T(\alpha_j \mathbf{z}_j) = \lambda_j \alpha_j \mathbf{z}_j.$$

But $\lambda_j \geq 0$, $\alpha_j \geq 0$, so $(\lambda_j \alpha_j) \geq 0$, so TU is positive semidefinite.