

Homework Set 7 Solutions

1.

- (a) (2 points) Give an example of an operator whose matrix with respect to some basis contains only zeroes on the diagonal, but the operator is invertible.

Solution. Let $T \in \mathcal{L}(\mathcal{R}^2)$ be given by

$$T(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

Since we have just switched entries, we should be able to undo this by applying T again. Hence we postulate that $T^{-1} = T$. Checking, we obtain

$$T^2(x_1, x_2) = T(x_2, x_1) = (x_1, x_2),$$

so the result is shown.

- (b) (2 points) Give an example of an operator whose matrix with respect to some basis contains only nonzero entries on the diagonal, but the operator is not invertible.

Solution. Let $T \in \mathcal{L}(\mathcal{R}^2)$ be given by

$$T(x_1, x_2) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -(x_1 + x_2) \end{pmatrix}.$$

Then we note that $T(1, -1) = \mathbf{0}$, so T is not invertible.

2.

- (a) (3 points) Recall that the *trace* of a matrix $A \in \mathcal{R}^{n \times n}$, denoted $\text{tr } A$, is defined as

$$\text{tr } A = \sum_{i=1}^n a_{ii}.$$

Let $B \in \mathcal{R}^{n \times n}$. Prove or disprove that the function $\langle A, B \rangle = \text{tr}(B^T A)$ is an inner product on $\mathcal{R}^{n \times n}$.

Solution. Calculating the function, we obtain

$$\begin{aligned} (B^T A)_{ij} &= \sum_{k=1}^n (B^T)_{ik} a_{kj} = \sum_{k=1}^n b_{ki} a_{kj} \\ \langle A, B \rangle &= \text{tr}(B^T A) = \sum_{i=1}^n (B^T A)_{ii} = \sum_{i=1}^n \sum_{k=1}^n b_{ki} a_{ki}. \end{aligned}$$

Checking the properties, we have that

$$\langle A, A \rangle = \sum_{i=1}^n \sum_{k=1}^n a_{ki}^2 \geq 0,$$

with $\langle A, A \rangle = 0$ if and only if $a_{ki} = 0$ for all k and i , which implies that $A = O$. So the first property checks. Checking the remaining properties, we have

$$\begin{aligned} \langle cA + D, B \rangle &= \sum_{i=1}^n \sum_{k=1}^n b_{ki}(cA + D)_{ki} = \sum_{i=1}^n \sum_{k=1}^n b_{ki}(ca_{ki} + d_{ki}) \\ &= c \sum_{i=1}^n \sum_{k=1}^n b_{ki}a_{ki} + \sum_{i=1}^n \sum_{k=1}^n b_{ki}d_{ki} = c \langle A, B \rangle + \langle D, B \rangle, \\ \langle A, B \rangle &= \sum_{i=1}^n \sum_{k=1}^n b_{ki}a_{ki} = \sum_{i=k}^n \sum_{k=1}^n a_{ki}b_{ki} = \langle B, A \rangle. \end{aligned}$$

Therefore, this function is an inner product.

- (b) (4 points) Let $T \in \mathcal{L}(V, W)$, V be a vector space over a field F , W be an inner-product space over F with a defined inner product $P_W(\mathbf{w}_1, \mathbf{w}_2)$ for $\mathbf{w}_i \in W$. Find conditions on T , if any, such that $P_V(\mathbf{v}_1, \mathbf{v}_2) = P_W(T(\mathbf{v}_1), T(\mathbf{v}_2))$ is an inner product on V .

Solution. Working through the properties in reverse, we obtain

$$P_V(\mathbf{v}_1, \mathbf{v}_2) = P_W(T(\mathbf{v}_1), T(\mathbf{v}_2)) = \overline{P_W(T(\mathbf{v}_2), T(\mathbf{v}_1))} = \overline{P_V(\mathbf{v}_2, \mathbf{v}_1)},$$

where we have used the fact that P_W is an inner product. Continuing to check, we have

$$\begin{aligned} P_V(c\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2) &= P_W(T(c\mathbf{v}_1 + \mathbf{v}_3), T(\mathbf{v}_2)) = P_W(cT(\mathbf{v}_1) + T(\mathbf{v}_3), T(\mathbf{v}_2)) \\ &= cP_W(T(\mathbf{v}_1), T(\mathbf{v}_2)) + P_W(T(\mathbf{v}_3), T(\mathbf{v}_2)) = cP_V(\mathbf{v}_1, \mathbf{v}_2) + P_V(\mathbf{v}_3, \mathbf{v}_2), \end{aligned}$$

where we have used the fact that T is linear. Lastly, we have

$$P_V(\mathbf{v}, \mathbf{v}) = P_W(T(\mathbf{v}), T(\mathbf{v})) \geq 0.$$

with equality if and only if $T(\mathbf{v}) = \mathbf{0}$. However, if T has a null space containing some $\mathbf{v} \neq \mathbf{0}$, then $P_V(\mathbf{v}, \mathbf{v}) = 0$ for that \mathbf{v} , which contradicts the definition of the inner product. Thus for P_V as defined to be an inner product, T must not have a nonzero null space, so T must be one-to-one.

3. For each of the following candidates, prove that the function is a norm on the given vector space V (listing any necessary conditions), or show that it violates some required property.

(a) (2 points) $V = C[0, 1]$, $\|f\| = \int_0^1 |f(t)| dt$

Solution. Clearly $\|f\| \geq 0$ with equality if and only if $f = 0$. Also,

$$\begin{aligned}\|cf\| &= \int_0^1 |cf(t)| dt = \int_0^1 |c| |f(t)| dt = |c| \|f\| \\ \|f + g\| &= \int_0^1 |f(t) + g(t)| dt \leq \int_0^1 |f(t)| + |g(t)| dt = \|f\| + \|g\|.\end{aligned}$$

Therefore, this is always a norm.

(b) (3 points) $V = \mathcal{P}_n$, $\|p\| = \sum_{i=0}^k p^2(x_i)$

Solution. This is not a norm since

$$\|ap\| = \sum_{i=0}^k (ap)^2(x_i) = a^2 \sum_{i=0}^k p^2(x_i) \neq |a| \|p\|.$$

(c) (2 points) $V = \mathcal{R}^{n \times n}$, $\|A\| = \max_{i,j} |a_{ij}|$

Solution. Clearly $\|A\| \geq 0$ with equality if and only if $A = O$. Also,

$$\begin{aligned}\|cA\| &= \max_{ij} |ca_{ij}| = |c| \max_{ij} |a_{ij}| = |c| \|A\| \\ \|A + B\| &= \max_{ij} |a_{ij} + b_{ij}| \leq \max_{ij} (|a_{ij}| + |b_{ij}|) \leq \max_{ij} |a_{ij}| + \max_{ij} |b_{ij}| = \|A\| + \|B\|.\end{aligned}$$

Therefore, this is always a norm.

4. Let $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$. Consider the following candidates for vector norms:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}, \quad p \geq 1; \quad \|\mathbf{x}\|_\infty = \max_i |x_i|.$$

(a) (5 points) Show that $\|\mathbf{x}\|_p$ is a vector norm by verifying the properties shown in class.

Solution. Clearly $\|\mathbf{x}\|_p \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$. Also,

$$\|c\mathbf{x}\|_p = \left(\sum_{i=1}^n |cx_i|^p \right)^{1/p} = \left(\sum_{i=1}^n |c|^p |x_i|^p \right)^{1/p} = |c| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} = |c| \|\mathbf{x}\|_p.$$

To prove the Triangle Inequality, we use Hölder's Inequality, which states that

$$\mathbf{x}^T \mathbf{y} \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q, \quad p^{-1} + q^{-1} = 1.$$

Note that if we solve for q , we have

$$q = \frac{p}{p-1}. \quad (\text{A})$$

Therefore, if $p > 1$, so is q and hence the norm is well-defined. However, q is not defined for $p = 1$, so we shall consider that case separately at the end. Note that this relates the product of the norms to the inner product, which is distributive. In particular, we note that if $\|\mathbf{y}\|_q = 1$,

$$\mathbf{x}^T \mathbf{y} \geq \|\mathbf{x}\|_p. \quad (\text{B})$$

We would like to construct a particular \mathbf{y} that satisfies the equality in (B). Hence, we must have that

$$\sum_{j=1}^n x_j y_j = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p}.$$

To match the sums, we let

$$y_j = \lambda \frac{|x_j|^p}{x_j}, \quad x_j \neq 0,$$

and $y_j = 0$ otherwise. Then we have

$$\begin{aligned} \lambda \sum_{j=1}^n x_j \frac{|x_j|^p}{x_j} &= \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \\ \lambda &= \left(\sum_{j=1}^n |x_j|^p \right)^{1/p-1} = \left(\sum_{j=1}^n |x_j|^p \right)^{-1/q}, \end{aligned}$$

where we have used (A).

Then for $\|\mathbf{y}\|_q = 1$, we must have that

$$\begin{aligned} \sum_{j=1}^n |y_j|^q &= 1 \\ \sum_{j=1}^n \left| \frac{|x_j|^p}{x_j} \right|^q &= \lambda^{-q} \\ \sum_{j=1}^n |x_j|^{(p-1)q} &= \sum_{j=1}^n |x_j|^p. \end{aligned}$$

Then again using (A), we see that (B) is satisfied with equality. Hence we have

$$\begin{aligned} \|\mathbf{x}\|_p &= \max_{\|\mathbf{y}\|_q=1} \mathbf{x}^T \mathbf{y} \\ \|\mathbf{x} + \mathbf{z}\|_p &= \max_{\|\mathbf{y}\|_q=1} (\mathbf{x} + \mathbf{z})^T \mathbf{y} \leq \max_{\|\mathbf{y}\|_q=1} \mathbf{x}^T \mathbf{y} + \max_{\|\mathbf{y}\|_q=1} \mathbf{z}^T \mathbf{y} = \|\mathbf{x}\|_p + \|\mathbf{z}\|_p, \end{aligned}$$

as required.

For the case $p = 1$, we have

$$\|\mathbf{x} + \mathbf{y}\|_1 = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n |x_i| + |y_i| = \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1,$$

so the Triangle Inequality is satisfied.

(b) (2 points) Show that

$$\lim_{p \rightarrow \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty.$$

Solution. WLOG let

$$x_1 = \max_i |x_i| = \|\mathbf{x}\|_\infty.$$

Then we have

$$\lim_{p \rightarrow \infty} \|\mathbf{x}\|_p = \lim_{p \rightarrow \infty} \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} = \lim_{p \rightarrow \infty} \left[|x_1|^p \left(1 + \sum_{i=2}^n \left| \frac{x_i}{x_1} \right|^p \right) \right]^{1/p} = |x_1| = \|\mathbf{x}\|_\infty,$$

where the result holds because $|x_i/x_1| \leq 1$ for all $i \neq 1$.

5. (3 points) Let $\mathbf{u}, \mathbf{v} \in V$. Prove that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ if and only if $\|\mathbf{u}\| \leq \|\mathbf{u} + a\mathbf{v}\|$ for every $a \in F$.

Solution. Since the norm is nonnegative, the inequality is equivalent to

$$\begin{aligned} \|\mathbf{u}\|^2 &\leq \|\mathbf{u} + a\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\Re(\langle \mathbf{u}, a\mathbf{v} \rangle) + \|a\mathbf{v}\|^2 \\ 0 &\leq 2\Re(a \langle \mathbf{u}, \mathbf{v} \rangle) + |a|^2 \|\mathbf{v}\|^2. \end{aligned}$$

Clearly if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, the inequality is satisfied. If the inequality is satisfied, it must hold for a and $-a$.

$$\begin{aligned} |a|^2 \|\mathbf{v}\|^2 &\geq -2\Re(a \langle \mathbf{u}, \mathbf{v} \rangle) \\ |-a|^2 \|\mathbf{v}\|^2 &\geq -2\Re(-a \langle \mathbf{u}, \mathbf{v} \rangle) \\ 0 &\geq -4\Re(a \langle \mathbf{u}, \mathbf{v} \rangle) \\ \Re(a \langle \mathbf{u}, \mathbf{v} \rangle) &\geq 0. \end{aligned} \tag{A}$$

Since (A) must hold for real a that are both positive and negative, we must have that $\Re(\langle \mathbf{u}, \mathbf{v} \rangle) = 0$. Now let $a = bi$, where b is real. Then (A) becomes

$$\Im(b \langle \mathbf{u}, \mathbf{v} \rangle) \geq 0,$$

which must be true for b that are both positive and negative, so $\Im(\langle \mathbf{u}, \mathbf{v} \rangle) = 0$. Hence $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

6. (4 points) Define an inner product on $C[-\pi, \pi]$ as follows:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) dx.$$

Show that the set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \mid k = 1, 2, \dots, n \right\}$$

is an orthonormal set.

Solution. For any nonzero integers j and k ,

$$\langle 1, \sin kx \rangle = 0, \quad \langle \sin jx, \cos kx \rangle = 0$$

since we are integrating an odd function over a symmetric domain. Similarly, for any nonzero integer k ,

$$\langle 1, \cos kx \rangle = \left[\frac{\sin kx}{k} \right]_{-\pi}^{\pi} = 0. \quad (\text{A})$$

Furthermore, since

$$\begin{aligned} \cos jx \cos kx &= \frac{\cos(j+k)x + \cos(j-k)x}{2}, \\ \sin jx \sin kx &= \frac{\cos(j-k)x - \cos(j+k)x}{2}, \end{aligned} \quad (\text{B})$$

we see that $\langle \cos jx, \cos kx \rangle$ and $\langle \sin jx, \sin kx \rangle$ can be written in the form (A) when $j \neq k$, and hence we have that these inner products are zero as well. Hence the set is orthogonal. To check the norms, we note that

$$\begin{aligned} \|1\|^2 = \langle 1, 1 \rangle = 2\pi &\implies \left\| \frac{1}{\sqrt{2\pi}} \right\|^2 = 1, \\ \|\cos kx\|^2 = \langle \cos kx, \cos kx \rangle = \frac{1}{2} \langle 1, 1 \rangle = \pi &\implies \left\| \frac{\cos kx}{\sqrt{\pi}} \right\|^2 = 1, \\ \|\sin kx\|^2 = \langle \sin kx, \sin kx \rangle = \frac{1}{2} \langle 1, 1 \rangle = \pi &\implies \left\| \frac{\sin kx}{\sqrt{\pi}} \right\|^2 = 1, \end{aligned}$$

as desired.

7. Let V be an inner product space and $W = \{\mathbf{w}_j\}_1^m$ be an orthonormal subset of V .

(a) (3 points) Prove *Bessel's Inequality*:

$$\|\mathbf{x}\|^2 \geq \sum_{j=1}^m |\langle \mathbf{x}, \mathbf{w}_j \rangle|^2.$$

Solution. Let $U = \text{Span } W$. By notes in class, we know that $\mathbf{x} \in V$ can be written uniquely as $\mathbf{x} = \text{proj}_U \mathbf{x} + \mathbf{z}$, where $\mathbf{z} \in U^\perp$. Since \mathbf{z} is orthogonal to $\text{proj}_U \mathbf{x}$, we may use the Pythagorean Theorem to yield

$$\begin{aligned} \|\mathbf{x}\|^2 &= \|\text{proj}_U \mathbf{x}\|^2 + \|\mathbf{z}\|^2 = \left\| \sum_{j=1}^m \langle \mathbf{x}, \mathbf{w}_j \rangle \mathbf{w}_j \right\|^2 + \|\mathbf{z}\|^2 = \sum_{j=1}^m |\langle \mathbf{x}, \mathbf{w}_j \rangle|^2 \|\mathbf{w}_j\|^2 + \|\mathbf{z}\|^2 \\ &= \sum_{j=1}^m |\langle \mathbf{x}, \mathbf{w}_j \rangle|^2 + \|\mathbf{z}\|^2 \geq \sum_{j=1}^m |\langle \mathbf{x}, \mathbf{w}_j \rangle|^2, \end{aligned}$$

where we have used the definition of $\text{proj}_U \mathbf{x}$ and the fact that W is an orthonormal set.

(b) (2 points) Prove that Bessel's Inequality becomes an equality if and only if $\mathbf{x} \in \text{Span } W$.

Solution. The inequality becomes an equality if and only if $\|\mathbf{z}\| = 0$, which implies that $\mathbf{z} = \mathbf{0}$ and $\mathbf{x} \in U$.

8. (3 points) Let U be a subspace of a finite-dimensional vector space V . Calculate proj_{U^\perp} in terms of proj_U .

Solution. By notes in class, we know that $\mathbf{x} \in V$ can be written uniquely as $\mathbf{x} = \text{proj}_U \mathbf{x} + \mathbf{z}$, where $\mathbf{z} \in U^\perp$. We know that $(U^\perp)^\perp = U$, so $\mathbf{x} \in V$ can be written uniquely as $\mathbf{x} = \mathbf{y} + \text{proj}_{U^\perp} \mathbf{x}$, where $\mathbf{y} \in U$. By the uniqueness, we have that $\mathbf{z} = \text{proj}_{U^\perp} \mathbf{x}$, so

$$\begin{aligned} \text{proj}_{U^\perp} \mathbf{x} &= \mathbf{x} - \text{proj}_U \mathbf{x} \\ \text{proj}_{U^\perp} &= I - \text{proj}_U. \end{aligned}$$