

## Homework Set 6 Solutions

1. (5 points) Suppose  $S, T \in \mathcal{L}(V)$  and  $S$  is invertible. Prove that if  $p(x)$  is a polynomial,  $p(STS^{-1}) = Sp(T)S^{-1}$ .

*Solution.* As a preliminary step, we prove that  $(STS^{-1})^n = ST^nS^{-1}$  for any  $n$ . We proceed by induction. Clearly it is true for  $n = 1$ . Assume it is true for  $n$ . Then

$$(STS^{-1})^{n+1} = (STS^{-1})^n(STS^{-1}) = ST^n(S^{-1}S)TS^{-1} = ST^{n+1}S^{-1},$$

as required. Then

$$p(STS^{-1}) = \sum_{j=0}^{\infty} a_j(STS^{-1})^j = \sum_{j=0}^{\infty} a_j ST^j S^{-1} = S \left( \sum_{j=0}^{\infty} a_j T^j \right) S^{-1} = Sp(T)S^{-1}.$$

2. (8 points) Two operators  $S$  and  $T \in \mathcal{L}(V)$  are said to be *simultaneously diagonalizable* if  $\mathcal{M}(T, Z, Z)$  and  $\mathcal{M}(S, Z, Z)$  are diagonal for the same basis  $Z$  of  $V$ . Prove that if  $S$  and  $T$  are simultaneously diagonalizable,  $ST = TS$ .

*Solution.* From notes in class, we know that  $Z = \{\mathbf{z}_j\}_1^n$  is an eigenvector basis for  $V$ . Denote the entries of  $\mathcal{M}(T, Z, Z)$  by  $t_{ij}$  and the entries of  $\mathcal{M}(S, Z, Z)$  by  $s_{ij}$ . Defining  $\mathbf{v} \in V$  by

$$\mathbf{v} = \sum_{j=1}^n c_j \mathbf{z}_j,$$

we obtain

$$\begin{aligned} (ST)\mathbf{v} &= S \left( \sum_{j=1}^n c_j T(\mathbf{z}_j) \right) = S \left( \sum_{j=1}^n c_j \sum_{i=1}^n t_{ij} \mathbf{z}_i \right) = S \left( \sum_{j=1}^n c_j t_{jj} \mathbf{z}_j \right) = \sum_{j=1}^n c_j t_{jj} S(\mathbf{z}_j) \\ &= \sum_{j=1}^n c_j t_{jj} \sum_{i=1}^n s_{ij} \mathbf{z}_i = \sum_{j=1}^n c_j t_{jj} s_{jj} \mathbf{z}_j, \end{aligned} \tag{A.1}$$

where we have used the fact that  $S$  and  $T$  are diagonal, so only  $s_{ii}$  and  $t_{ii}$  are nonzero. Similarly,

$$\begin{aligned} (TS)\mathbf{v} &= \left( \sum_{j=1}^n c_j S(\mathbf{z}_j) \right) = T \left( \sum_{j=1}^n c_j \sum_{i=1}^n s_{ij} \mathbf{z}_i \right) = T \left( \sum_{j=1}^n c_j s_{jj} \mathbf{z}_j \right) = \sum_{j=1}^n c_j s_{jj} T(\mathbf{z}_j) \\ &= \sum_{j=1}^n c_j s_{jj} \sum_{i=1}^n t_{ij} \mathbf{z}_i = \sum_{j=1}^n c_j s_{jj} t_{jj} \mathbf{z}_j. \end{aligned} \tag{A.2}$$

Since equations (A) agree and  $\mathbf{v}$  is arbitrary,  $ST = TS$ .

3. (5 points) Let  $T \in \mathcal{L}(V)$  be an invertible operator such that  $\mathcal{M}(T, Z, Z)$  is diagonal for some basis  $Z$  of  $V$ . Find a basis  $B$  of  $V$  such that  $\mathcal{M}(T^{-1}, B, B)$  is also diagonal. If the diagonal entries of  $\mathcal{M}(T, Z, Z)$  are given by  $t_{ii}$ , find the diagonal entries of  $\mathcal{M}(T^{-1}, B, B)$ .

*Solution.* We note that if  $T\mathbf{z} = \lambda\mathbf{z}$  and  $T$  is invertible, we have

$$\begin{aligned}\mathbf{z} &= T^{-1}(T\mathbf{z}) = T^{-1}(\lambda\mathbf{z}) = \lambda(T^{-1}\mathbf{z}) \\ \lambda(T^{-1}\mathbf{z}) - \lambda^{-1}\mathbf{z} &= \mathbf{0},\end{aligned}$$

so  $\lambda^{-1}$  is an eigenvalue for  $T^{-1}$  corresponding to  $\mathbf{z}$ . Therefore eigenvectors of  $T^{-1}$  are eigenvectors of  $T$ , and hence  $\mathcal{M}(T^{-1}, Z, Z)$  is diagonal. Letting the diagonal entries of  $\mathcal{M}(T^{-1}, Z, Z)$  be given by  $s_{ii}$ , we see that  $s_{ii} = t_{ii}^{-1}$ , since the eigenvalues are reciprocals of one another. (We see that since  $T$  is invertible, it has no zero eigenvalues, so  $t_{ii}^{-1}$  always exists.)

4. (8 points) Let  $T \in \mathcal{L}(V)$ ,  $\dim V = n$ ,  $\lambda$  be an eigenvalue of  $T$ . Furthermore, assume that the dimension of the eigenspace of  $T$  corresponding to  $\lambda$  is  $m$ . Show that there exists a basis  $W$  for  $V$  such that

$$\mathcal{M}(T, W, W) = \begin{pmatrix} \lambda I & B \\ O & C \end{pmatrix}, \quad B \in \mathcal{R}^{m \times (n-m)}, \quad C \in \mathcal{R}^{(n-m) \times (n-m)},$$

where  $I$  is the identity matrix in  $\mathcal{R}^{m \times m}$ , and  $O$  is the zero matrix in  $\mathcal{R}^{(n-m) \times m}$ .

*Solution.* Let  $B_W = \{\mathbf{w}_j\}_1^m$  be a basis for the eigenspace of  $T$  corresponding to  $\lambda$ . Extend it to a basis  $W$  for  $V$  by adding  $S = \{\mathbf{s}_j\}_{m+1}^n$ . Then since  $T(\mathbf{w}_j) = \lambda\mathbf{w}_j$ , we have

$$T(\mathbf{w}_j) = \sum_{i=1}^m m_{ij} \mathbf{w}_i + \sum_{i=m+1}^n m_{ij} \mathbf{s}_i = \lambda \mathbf{w}_j + \sum_{\substack{i=1 \\ i \neq j}}^m 0 \mathbf{w}_i + \sum_{i=m+1}^n 0 \mathbf{s}_i,$$

where  $m_{ij}$  is the  $ij$ th entry of  $\mathcal{M}(T, W, W)$ . Since  $m_{ij} = 0$  for  $i \neq j$  in the first  $m$  columns of  $\mathcal{M}(T, W, W)$  and  $m_{ii} = \lambda$ , these columns form

$$\begin{pmatrix} \lambda I \\ O \end{pmatrix},$$

where  $I$  is the identity matrix in  $\mathcal{R}^{m \times m}$ , and  $O$  is the zero matrix in  $\mathcal{R}^{n-m \times m}$ . The other entries are arbitrary, so

$$\mathcal{M}(T, W, W) = \begin{pmatrix} \lambda I & B \\ O & C \end{pmatrix}, \quad B \in \mathcal{R}^{m \times (n-m)}, \quad C \in \mathcal{R}^{(n-m) \times (n-m)},$$

as desired.

5. (10 points) Let  $T \in \mathcal{L}(V)$ ,  $V$  a vector space over  $\mathcal{R}$ ,  $p(x)$  a polynomial in  $\mathcal{R}$ , and  $a \in \mathcal{R}$ . Show that the following statement is false:  
 $a$  is an eigenvalue of  $p(T)$  if and only if  $a = p(\lambda)$  for some eigenvalue  $\lambda$  of  $T$ .

Rather than simply providing a counterexample, prove as strong a statement as possible.

*Solution.* We begin by proving the statement in the reverse direction. Let  $\lambda \in \mathcal{R}$  be an eigenvalue of  $T$ , so  $T\mathbf{z} = \lambda\mathbf{z}$  for some vector  $\mathbf{z} \in V$ . Moreover, let

$$p(T) = \sum_{j=0}^n c_j T^j, \quad a = p(\lambda).$$

Then

$$p(T)\mathbf{z} = \sum_{j=0}^n c_j T^j \mathbf{z} = \sum_{j=0}^n c_j \lambda^j \mathbf{z} = p(\lambda)\mathbf{z} = a\mathbf{z}.$$

Thus  $a$  is an eigenvalue of  $p(T)$ . Now suppose that  $p(T)\mathbf{z} = a\mathbf{z}$ . The converse is clearly false, since an operator on a vector space over  $\mathcal{R}$  is not guaranteed to have an eigenvalue. By factoring as we did in class, we have that

$$\begin{aligned} & [p(T) - a]\mathbf{z} = \mathbf{0} \\ & \left[ c_n \prod_{i=1}^r (T - \delta_i I) \prod_{i=1}^c (T^2 + b_i T + d_i I) \right] \mathbf{z} = \mathbf{0}, \end{aligned}$$

where  $\delta_i$  are the eigenvalues of  $T$ . So  $\mathbf{z}$  is in the null space of one of these operators. If  $\mathbf{z} \in \mathcal{N}(T - \delta_i I)$ , then  $\mathbf{z}$  is an eigenvector for  $T$  with eigenvalue  $\delta_i$ , and we have

$$p(T)\mathbf{z} = \sum_{j=0}^n c_j T^j \mathbf{z} = \sum_{j=0}^n c_j \delta_i^j \mathbf{z} = p(\delta_i)\mathbf{z} = a\mathbf{z}, \quad (\text{A})$$

so  $a = p(\delta_i)$ . Thus if  $T$  has all real eigenvalues, the result holds. Note that equation (A) holds even if  $\delta_i$  is complex. Thus if  $T$  (as considered as an operator over a vector space over  $\mathcal{C}$ ) has even one complex eigenvalue, then the result does not hold, because it is possible that  $a = p(\delta_i)$  for that complex eigenvalue.

6. (4 points) Suppose  $V$  is a real vector space and  $T \in \mathcal{L}(V)$  has no eigenvalues. Prove that every subspace of  $V$  invariant under  $T$  has even dimension.

*Solution.* Proof by contradiction. Suppose  $W$  is an odd-dimensional invariant subspace of  $T$ . Then  $W$  is also invariant under  $T|_W$ , so by Theorem 5.26,  $T|_W$  has an eigenvalue, so  $T|_W \mathbf{z} = \lambda\mathbf{z}$  for some  $\mathbf{z} \in W$ . But  $T|_W \mathbf{z} = T\mathbf{z}$  for all  $\mathbf{z} \in W$ , so  $T\mathbf{z} = \lambda\mathbf{z}$ , which is a contradiction.