Homework Set 5 Solutions

1. (4 points) Suppose that $B_V = \{\mathbf{v}_j\}_1^n$ is a basis for V. Prove that the function $T : V \to \mathcal{F}^{n\times 1}$ defined by $T(\mathbf{v}) = [\mathbf{v}]_{B_V}$ is an invertible linear map of V onto $\mathcal{F}^{n\times 1}$.

Solution. First, we prove that T is linear. Let

$$\mathbf{v} = \sum_{j=1}^{n} a_j \mathbf{v}_j, \quad \mathbf{w} \sum_{j=1}^{n} b_j \mathbf{v}_j.$$

Then

$$\begin{aligned} T(c\mathbf{v}+\mathbf{w}) &= [c\mathbf{v}+\mathbf{w}]_{B_V} = \begin{pmatrix} ca_1+b_1 \\ ca_2+b_2 \\ \vdots \\ ca_n+b_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = c[\mathbf{v}]_{B_V} + [\mathbf{w}]_{B_V} \\ &= cT(\mathbf{v}) + T(\mathbf{w}), \end{aligned}$$

so T is linear, as required. To prove the fact that T is one-to-one, we note that

$$T(\mathbf{v}) = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{v} = \sum_{j=1}^{n} 0\mathbf{v}_j = \mathbf{0},$$

so T is one-to-one. To prove that T is onto, let $\mathbf{a} \in \mathcal{F}^{n\times 1}$ be defined by

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Clearly any $\mathbf{a} \in \mathcal{F}^{n\times 1}$ can be written this way. But $\mathbf{a} = [\mathbf{v}]_{B_V}$, where $\mathbf{v}$ is defined above, so T is onto. Since T is one-to-one and onto, it must be invertible.

2. Consider the following transformations:

$$T_1 \in \mathcal{L}\left(P_2 \times \mathcal{R}, \mathcal{R}^2 \times P_1\right), T_1\left(p(x), y\right) = (y + p(1), p(0) - 2y, p'(x))$$

$$T_2 \in \mathcal{L}(\mathcal{R}^2 \times P_1, \mathcal{R}^3), \quad T_2(z_1, z_2, p(x)) = \left(p'(1) - 3z_1, \int_0^1 xp(x)\,dx, z_1 + p(0)\right),$$

where the notation $V \times W$ indicates the vector space of ordered pairs where the first element is in V and the second is in W. Verify the following facts. (Use the standard bases E in all cases, with powers increasing for polynomial vector spaces.)

(a) (5 points) $\mathcal{M}(T_2T_1) = \mathcal{M}(T_2)\,\mathcal{M}(T_1)$

Solution. We construct each matrix by finding the effect of the transformation on the basis vectors:

$$T_1(1,0) = \left(0+1, 1-2(0), \frac{d(1)}{dx}\right) = \mathbf{e}_1 + \mathbf{e}_2 + 0(1) + 0(x)$$
$$T_1(x,0) = \left(0+1, 0-2(0), \frac{d(x)}{dx}\right) = \mathbf{e}_1 + 0\mathbf{e}_2 + 1(1) + 0(x)$$
$$T_1(x^2,0) = \left(0+1, 0-2(0), \frac{d(x)^2}{dx}\right) = \mathbf{e}_1 + 0\mathbf{e}_2 + 0(1) + 2(x)$$
$$T_1(0,1) = \left(1+0, 0-2(1), \frac{d(0)}{dx}\right) = \mathbf{e}_1 - 2\mathbf{e}_2 + 0(1) + 0(x)$$
$$\mathcal{M}(T_1) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$
$$T_2(1,0,0) = \left(0-3(1), \int_0^1 0\,dx, 1+0\right) = -3\mathbf{e}_1 + 0\mathbf{e}_2 + \mathbf{e}_3$$
$$T_2(0,1,0) = \left(0-3(0), \int_0^1 0\,dx, 0+0\right) = 0\mathbf{e}_1 + 0\mathbf{e}_2 + \mathbf{e}_3$$
$$T_2(0,0,1) = \left(0-3(0), \int_0^1 x\,dx, 0+1\right) = 0\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \mathbf{e}_3$$
$$T_2(0,0,x) = \left(1-3(0), \int_0^1 x^2\,dx, 0+0\right) = 1\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 + 0\mathbf{e}_3$$
$$\mathcal{M}(T_2) = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 1/3 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$
$$T_2T_1(1,0) = T_2(1,1,0) = \left(0-3(1), \int_0^1 0\,dx, 1+0\right) = -3\mathbf{e}_1 + 0\mathbf{e}_2 + \mathbf{e}_3$$
$$T_2T_1(x,0) = T_2(1,0,1) = \left(0-3(1), \int_0^1 x\,dx, 1+1\right) = -3\mathbf{e}_1 + \frac{1}{2}\mathbf{e}2 + 2\mathbf{e}_3$$
$$T_2T_1(x^2,0) = T_2(1,0,2x) = \left(2-3(1), \int_0^1 2x^2\,dx, 1+0\right) = -1\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2 + \mathbf{e}_3$$
$$T_2T_1(0,1) = T_2(1,-2,0) = \left(0-3(1), \int_0^1 0\,dx, 1+0\right) = -3\mathbf{e}_1 + 0\mathbf{e}_2 + \mathbf{e}_3$$

$$\mathcal{M}(T_2T_1) = \begin{pmatrix} -3 & -3 & -1 & -3 \\ 0 & 1/2 & 2/3 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 1/3 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

(b) (2 points) $\left[T_1(x^2-1,3)\right]_E = \mathcal{M}(T_1)\left[(x^2-1,3)\right]_E$

Solution. We construct each matrix by finding the effect of the transformation on the basis vectors:

$$\begin{aligned} T_1(x^2-1,3) &= \left(3+0,\ -1-2(3), \frac{d(x^2-1)}{dx}\right) = 3\mathbf{e}_1 - 7\mathbf{e}_2 + 0(1) + 2(x) \\ [T_1(x^2-1,3)]_E &= (3,-7,0,2) \\ \mathcal{M}(T_1)[(x^2-1,3)]_E &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

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(c) (2 points) $\left[T_2T_1(x^2-1,3)\right]_E = \mathcal{M}(T_2)\left[T_1(x^2-1,3)\right]_E$

Solution. We construct each matrix by finding the effect of the transformation on the basis vectors:

$$\begin{aligned} T_2T_1(x^2-1,3) &= T_2(3,-7,2x) = \left(2-3(3), \int_0^1 2x^2\,dx, 3+0\right) \\ &= -7\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2 + 3\mathbf{e}_3 \\ [T_2T_1(x^2-1,3)]_E &= (-7,2/3,3) \\ \mathcal{M}(T_2)[T_1(x^2-1,3)]_E &= \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 1/3 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ 2/3 \\ 3 \end{pmatrix}. \end{aligned}$$

3. Let $S, T \in \mathcal{L}(V)$, V finite-dimensional.

(a) (5 points) Prove that ST and TS have the same eigenvalues.

Solution. Let $\mathbf{z}$ be an eigenvector for ST corresponding to λ. Then

$$(TS)T\mathbf{z} = T(ST\mathbf{z}) = T(\lambda z) = \lambda(T\mathbf{z}),$$

so $T\mathbf{z}$ is an eigenvector for TS if $T\mathbf{z} \neq \mathbf{0}$. If $T\mathbf{z} = \mathbf{0}$, we see that $\lambda = 0$ must be an eigenvalue for ST because $ST\mathbf{z} = S\mathbf{0} = \mathbf{0} = 0\mathbf{z}$. If $\mathcal{N}(S) \neq \{\mathbf{0}\}$, then for any $\mathbf{y} \in \mathcal{N}(S)$, we have $TS\mathbf{y} = T\mathbf{0} = \mathbf{0} = 0\mathbf{y}$, so $\lambda = 0$ is an eigenvalue for TS. If S is invertible, let $\mathbf{y} = S^{-1}\mathbf{z}$. Then we have $TS\mathbf{y} = T\mathbf{z} = \mathbf{0} = 0\mathbf{y}$, so $\lambda = 0$ is an eigenvalue for TS.

(b) (3 points) Show that $ST - TS \neq I$ for any $S, T \in \mathcal{L}(V)$.

Solution. Assume that $ST - TS = I$, and let $\mathbf{z}$ be an eigenvector for ST corresponding to λ, the smallest eigenvalue for ST (and hence the smallest eigenvalue for TS). Then

$$\begin{aligned}(ST - TS)\mathbf{z} &= I\mathbf{z} \\ (TS)\mathbf{z} &= \lambda\mathbf{z} - \mathbf{z},\end{aligned}$$

and hence we see that $\mathbf{z}$ is an eigenvector for TS corresponding to $\lambda - 1$. But this contradicts λ being the smallest eigenvalue, so $ST - TS \neq I$.

4. (4 points) Let $T \in \mathcal{L}(V)$, U a T-invariant subspace of V. Let $\dim U = m, \dim V = n$. Show that there exists a basis W for V such that

$$\mathcal{M}(T, W, W) = \begin{pmatrix} A & B \\ O & C \end{pmatrix}, \ A \in \mathcal{R}^{m\times m}, \ B \in \mathcal{R}^{m\times(n-m)}, \ C \in \mathcal{R}^{(n-m)\times(n-m)},$$

and O is the zero matrix in $\mathcal{R}^{(n-m)\times m}$.

Solution. Let $W = \{\mathbf{w}_j\}_1^m$ be a basis for U. Extend it to a basis B_V for V by adding $S = \{\mathbf{s}_j\}_{m+1}^n$. Then since $T(\mathbf{w}_j) \in U$, we have

$$T(\mathbf{w}_j) = \sum_{i=1}^m m_{ij}\mathbf{w}_i + \sum_{i=m+1}^n m_{ij}\mathbf{s}_i = \sum_{i=1}^m m_{ij}\mathbf{w}_i + \sum_{i=m+1}^n 0\mathbf{s}_i,$$

where m_{ij} is the ijth entry of $\mathcal{M}(T, W, W)$. Thus the last $n - m$ entries of the first m columns of $\mathcal{M}(T, W, W)$ are zero, and so form $O \in \mathcal{R}^{(n-m)\times m}$. The other entries are arbitrary, so

$$\mathcal{M}(T, W, W) = \begin{pmatrix} A & B \\ O & C \end{pmatrix}, \ A \in \mathcal{R}^{m\times m}, \ B \in \mathcal{R}^{m\times(n-m)}, \ C \in \mathcal{R}^{(n-m)\times(n-m)},$$

as desired.

5. An operator $N \in \mathcal{L}(V)$ is called *nilpotent* if $N^k = O$ for some integer k.

(a) (3 points) Show that the only eigenvalue for N is $\lambda = 0$.

Solution. Let $\mathbf{z}$ be an eigenvector for N. Then

$$\begin{array}{ccccc} N\mathbf{z} & = & \lambda\mathbf{z} & & \\ N^2\mathbf{z} = N(N\mathbf{z}) & = & N(\lambda\mathbf{z}) = \lambda N\mathbf{z} = \lambda^2\mathbf{z} & & \\ N^k\mathbf{z} & = & 0\mathbf{z} = \mathbf{0} & = & \lambda^k\mathbf{z}. \end{array}$$

Since $\mathbf{z} \neq \mathbf{0}$, $\lambda^k = 0$ and hence $\lambda = 0$.

(b) (3 points) Let $V = \mathcal{R}^n$. Give an example of a nilpotent operator N such that $N^n = O$, but $N^j \neq O$ for any $j < n$.

Solution. Let

$$N(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, 0).$$

Clearly $N \neq 0$. Applying N is equivalent to shifting all the entries to the left, dropping the first and adding a zero to the end. Thus by applying N repeatedly, we have

$$\begin{aligned} N^j(x_1, x_2, \ldots, x_n) &= (x_{j+1}, x_{j+2}, \ldots, x_n, 0, \ldots, 0) \\ N^{n-1}(x_1, x_2, \ldots, x_n) &= (x_n, 0, \ldots, 0) \\ N^n(x_1, x_2, \ldots, x_n) &= (0, 0, \ldots, 0). \end{aligned}$$

Therefore, we see that $N^j \neq 0$ for any $j < n$, but also $N^n = 0$.

6. (3 points) Let $T \in \mathcal{L}(V)$ be an invertible transformation, and let U be a subspace of V. Prove that if U is invariant under T, U is invariant under T^{-1} .

Solution. Let $B_U = \{\mathbf{u}_i\}_1^n$ be a basis for U, and let $\mathbf{v} \in U$ be defined by

$$\mathbf{v} = \sum_{i=1}^n c_i \mathbf{u}_i.$$

If U is invariant under T, then

$$T(\mathbf{v}) = T\left(\sum_{i=1}^n c_i \mathbf{u}_i\right) = \sum_{i=1}^n c_i T(\mathbf{u}_i) = \sum_{i=1}^n d_i \mathbf{u}_i$$

for some constants d_i. Since T is invertible, it is onto, so every $\mathbf{u} \in U$ can be written as

$$\mathbf{u} = \sum_{i=1}^n d_i \mathbf{u}_i.$$

But then

$$T^{-1}(\mathbf{u}) = T^{-1}\left(\sum_{i=1}^n d_i \mathbf{u}_i\right) = \sum_{i=1}^n T^{-1}(c_i T(\mathbf{u}_i)) = \sum_{i=1}^n c_i \mathbf{u}_i \in U,$$

and hence U is invariant under T^{-1} as well.

7. Let $T \in \mathcal{L}(V)$,

$$V = \bigoplus_{i=1}^n U_i,$$

where U_i is invariant under T. Show that

(a) (3 points)

$$R(T) = \bigoplus_{i=1}^n R\left(T\big|_{U_i}\right)$$

Solution. Since

$$V = \bigoplus_{i=1}^n U_i,$$

each element $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{u}_i, \ \mathbf{u}_i \in U_i, \tag{A}$$

for a *unique* choice of vectors $\mathbf{u}_i$, so every $T\mathbf{v} \in R(T)$ can be written as

$$T\mathbf{v} = \sum_{i=1}^{n} T\mathbf{u}_i,$$

Since the U_i are invariant under T, $T\mathbf{u}_i \in U_i$, and hence $T\mathbf{u}_i \in R(T\big|_{U_i})$. Moreover, since

$$V = \bigoplus_{i=1}^{n} U_i,$$

we see that $T\mathbf{u}_i \neq T\mathbf{u}_j$ for any $i \neq j$, so the representation is unique. Since every vector in $R(T)$ can be written uniquely as a sum of vectors in $R(T\big|_{U_i})$,

$$R(T) = \bigoplus_{i=1}^{n} R(T\big|_{U_i}).$$

(b) (3 points)

$$\mathcal{N}(T) = \bigoplus_{i=1}^{n} \mathcal{N}\left(T\big|_{U_i}\right)$$

Solution. Let $\mathbf{v} \in \mathcal{N}(T)$. Applying T to both sides of (A), we obtain

$$T\mathbf{v} = \mathbf{0} = \sum_{i=1}^{n} T\mathbf{u}_i = \sum_{i=1}^{n} \mathbf{w}_i, \quad \mathbf{w}_i \in U_i,$$

since the U_i are invariant under T. But then by (A) with $\mathbf{v} = \mathbf{0}$, we see that $\mathbf{w}_i = \mathbf{0}$ by the definition of the direct sum. Thus $T\mathbf{u}_i = \mathbf{0}$, which implies that $\mathbf{u}_i \in \mathcal{N}(T\big|_{U_i})$. Since every $\mathbf{v} \in \mathcal{N}(T)$ can be written uniquely as a sum of these $\mathbf{u}_i$, we have that

$$\mathcal{N}(T) = \bigoplus_{i=1}^{n} \mathcal{N}(T\big|_{U_i}).$$