MATH 672-010 Vector Spaces
Prof. D. A. Edwards Due: Oct. 6, 2008

Homework Set 4 Solutions

1. Let $T \in \mathcal{L}(V, W)$, $S_V = \{\mathbf{s}_i\}_1^n \in V$, $S_W = \{T(s_i)\}_1^n \in W$. For each of the following statements, provide a proof or counterexample.

(a) (2 points) If S_V is linearly independent, so is S_W.

Solution. This is false. For instance, let T be the zero transformation. Then $S_W = \{\mathbf{0}\}$, which is linearly dependent.

(b) (2 points) If S_W is linearly independent, so is S_V.

Solution. This is true. Suppose S_V is linearly dependent. Then

$$\sum_{i=1}^{n} c_i \mathbf{s}_i = \mathbf{0}$$

for not all of the $c_i = 0$. Then applying the transformation, we have

$$T\left(\sum_{i=1}^{n} c_i \mathbf{s}_i\right) = \sum_{i=1}^{n} c_i T(\mathbf{s}_i) = \mathbf{0}$$

for not all of the $c_i = 0$. But this contradicts S_W being linearly independent, so S_V must be linearly independent.

2. (5 points) Let U and V be finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \mathcal{N}(ST) \leq \dim \mathcal{N}(S) + \dim \mathcal{N}(T).$$

Solution. Define a new map $S_* \in \mathcal{L}(R(T), W)$ to be the same as S where both are defined:

$$S_*\mathbf{v} = S\mathbf{v} \quad \forall \mathbf{v} \in R(T) \subseteq V.$$

We note that

$$R(S_*) = \{S\mathbf{v} \in W \mid \mathbf{v} = T\mathbf{u} \text{ for some } \mathbf{u} \in U\} = \{(ST)\mathbf{u} \text{ for some } \mathbf{u} \in U\} = R(ST),$$

so from Theorem 3.4 we have that

$$\dim R(T) = \dim \mathcal{N}(S_*) + \dim R(S_*) = \dim \mathcal{N}(S_*) + \dim R(ST). \tag{A}$$

T and ST are both maps from U, so we have from Theorem 3.4 that

$$\begin{aligned} \dim U = \dim \mathcal{N}(T) + \dim\ R(T) &= \dim \mathcal{N}(ST) + \dim R(ST) \\ \dim \mathcal{N}(T) + \dim \mathcal{N}(S_*) &= \dim\ \mathcal{N}(ST), \end{aligned}$$

where we have used (A). Since there may exist $\mathbf{v} \in V$, $\mathbf{v} \notin R(T)$ such that $S\mathbf{v} = \mathbf{0}$, we see that $\mathcal{N}(S_*) \subseteq \mathcal{N}(S)$, so $\dim \mathcal{N}(S_*) \leq \dim\ \mathcal{N}(S)$. Thus

$$\dim \mathcal{N}(ST) \leq \dim \mathcal{N}(T) + \dim \mathcal{N}(S),$$

as required.

3. For any polynomial $p(t) \in \mathcal{P}_3$, let $D(p) = p'$.
   (a) (2 points) Show that if we consider $D : \mathcal{P}_3 \to \mathcal{P}_2$, D is onto, but not one-to-one. In the process, calculate the dimensions necessary to verify Theorem 3.4.

Solution.

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

Since there are three undetermined constants in the range, we see that $\dim R(D) = 3 = \dim \mathcal{P}_2$, so D is onto. However, the derivative of any constant polynomial is zero, so D is not one-to-one. $\mathcal{N}(D) = \text{Span}\{a_0\}$, so $\dim \mathcal{N}(D) = 1$. Thus Theorem 3.4 checks, since

$$\begin{array}{ccl} \dim \mathcal{P}_3 & = & \dim \mathcal{N}(D) + \dim R(D) \\ 4 & = & 1 + 3. \end{array}$$

Now let $S = \{p \in \mathcal{P}_3 \,|\, p(1) = 0\}$.

   (b) (2 points) Show that if we consider $D : S \to \mathcal{P}_3$, D is one-to-one, but not onto. In the process, calculate the dimensions necessary to verify Theorem 3.4.

Solution. A basis for S is given by $\{(x-1)^i\}_1^3$, and so $\dim S = 3$. Thus we have that

$$D(a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3) = a_1 + 2a_2(x-1) + 3a_3(x-1)^2.$$

Since there are three undetermined constants in the range, we see that $\dim R(D) = 3 < \dim \mathcal{P}_3$, so D is not onto. Since the constant polynomials are not in S, we see from part (a) that $\mathcal{N}(D) = \mathbf{0}$, so $\dim \mathcal{N}(D) = 0$. Thus Theorem 3.4 checks, since

$$\begin{array}{ccl} \dim S & = & \dim \mathcal{N}(D) + \dim R(D) \\ 3 & = & 0 + 3. \end{array}$$

4. Consider the following transformation $T : \mathcal{R}^3 \to \mathcal{R}^3$:

$$T(x_1, x_2, x_3) = (x_2, x_2, x_2).$$

   (a) (2 points) Calculate $\mathcal{N}(T)$.

Solution. $T(x) = \mathbf{0}$ implies $x_2 = 0$, so $\mathcal{N}(T) = \{(x_1, 0, x_3)\}$.

   (b) (2 points) Calculate $R(T)$.

Solution. By construction, $R(T) = \{(x_2, x_2, x_2)\} = \text{Span}(1, 1, 1)$.

   (c) (2 points) Verify Theorem 3.4.

Solution. By part (a), we have that $\dim \mathcal{N}(T) = 2$. By part (b), we have that $\dim R(T) = 1$. Thus Theorem 3.4 checks, since

$$\begin{array}{ccc} \dim \mathcal{R}^3 & = & \dim \mathcal{N}(T) + \dim R(T) \\ 3 & = & 2 + 1. \end{array}$$

(d) (2 points) Is T one-to-one? Onto? Invertible?

Solution. T is not one-to-one because $\dim \mathcal{N}(T) > 0$, so it is not invertible. It is not onto because $\dim R(T) < 3$.

5. Consider the following transformation $T \in \mathcal{L}(V, W)$:

$$f \in C^2(\mathcal{R}) : \quad T(f) = t^2 f'' - 6tf' + 12f.$$

(a) (4 points) Let $V = W = C^2(\mathcal{R})$. Calculate $\mathcal{N}(T)$. Is T one-to-one? Invertible?

Solution. $T(f) = 0$ is an Euler equation, so the solution is given by $f = t^\lambda$. Substituting this in, we have

$$\begin{array}{ccc} t^2[\lambda(\lambda-1)t^{\lambda-2}] - 6t(\lambda t^{\lambda-1}) + 12t^\lambda & = & 0 \\ \lambda^2 - 7\lambda + 12 = (\lambda-4)(\lambda-3) & = & 0 \\ T(c_1 t^4 + c_2 t^3) & = & 0, \end{array}$$

and hence $\mathcal{N}(T) = \text{Span}\{t^4, t^3\}$. T is neither one-to-one nor invertible because it has a nontrivial null space.

(b) (3 points) Let $V = W = \mathcal{P}_2(\mathcal{R})$. Calculate $\mathcal{N}(T)$ and $R(T)$. Verify Theorem 3.4 and determine whether T is one-to-one, onto, or invertible.

Solution. Since t^4 and t^3 are not in V, we see that when restricted to these vector spaces, $\mathcal{N}(T) = \mathbf{0}$, so T is one-to-one. To calculate the range, we note that

$$\begin{array}{ccl} T(a_0 + a_1 t + a_2 t^2) & = & t^2(2a_2) - 6t(a_1 + 2a_2 t) + 12(a_0 + a_1 t + a_2 t^2) \\ & = & 2a_2 t^2 + 6a_1 t + 12a_0. \end{array}$$

Since there are three degrees of freedom on the right-hand side, we see that $R(T) = \mathcal{P}_2$. Thus T is also onto, so it is invertible. Also, Theorem 3.4 checks, since

$$\begin{array}{ccc} \dim \mathcal{P}_2 & = & \dim \mathcal{N}(T) + \dim R(T) \\ 3 & = & 0 + 3. \end{array}$$

6. Let $B_1 = \{\cos x, \sin x\}$ and $B_2 = \{\cos x + \sin x, \cos x - \sin x\}$ be two bases for the same subspace V of $C^1(\mathcal{R})$, and let $D \in \mathcal{L}(V)$ be defined by $D(f) = f'$ for any $f \in V$. Compute the following:

(a) (2 points) $\mathcal{M}(D, B_1, B_1)$

Solution. Taking the transformation of each of the basis vectors in the domain, we have

$$\begin{array}{rcc} D(\cos x) = -\sin x = a_{11}(\cos x) + a_{21}(\sin x) & \Longrightarrow & a_{11} = 0,\ a_{21} = -1 \\ D(\sin x) = \cos x = a_{12}(\cos x) + a_{22}(\sin x) & \Longrightarrow & a_{12} = 1,\ a_{22} = 0 \end{array}$$

$$\mathcal{M}(D, B_1, B_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(b) (2 points) $\mathcal{M}(D, B_1, B_2)$

Solution. Taking the transformation of each of the basis vectors in the domain, we have

$$\begin{array}{rcc} D(\cos x) = -\sin x = a_{11}(\cos x + \sin x) + a_{21}(\cos x - \sin x) & \Longrightarrow & a_{11} = -\frac{1}{2},\ a_{21} = \frac{1}{2} \\ D(\sin x) = \cos x = a_{12}(\cos x + \sin x) + a_{22}(\cos x - \sin x) & \Longrightarrow & a_{12} = \frac{1}{2},\ a_{22} = \frac{1}{2} \end{array}$$

$$\mathcal{M}(D, B_1, B_2) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

(c) (2 points) $\mathcal{M}(D, B_2, B_1)$

Solution. Taking the transformation of each of the basis vectors in the domain, we have

$$\begin{array}{rcc} D(\cos x + \sin x) = \sin x + \cos x = a_{11}(\cos x) + a_{21}(\sin x) & \Longrightarrow & a_{11} = 1,\ a_{21} = -1 \\ D(\cos x - \sin x) = -\sin x - \cos x = a_{12}(\cos x) + a_{22}(\sin x) & \Longrightarrow & a_{12} = -1,\ a_{22} = -1 \end{array}$$

$$\mathcal{M}(D, B_2, B_1) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

(d) (2 points) $\mathcal{M}(D, B_2, B_2)$

Solution. Taking the transformation of each of the basis vectors in the domain, we have

$$\begin{array}{rcl} D(\cos x + \sin x) & = & -\sin x + \cos x = a_{11}(\cos x + \sin x) + a_{21}(\cos x - \sin x) \\ a_{11} & = & 0,\ a_{21} = 1 \\ D(\cos x - \sin x) & = & -\sin x - \cos x = a_{12}(\cos x + \sin x) + a_{22}(\cos x - \sin x) \\ a_{12} & = & -1,\ a_{22} = 0 \end{array}$$

$$\mathcal{M}(D, B_2, B_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

7. Let $T : \mathcal{P}_2 \to \mathcal{R}^2$ be defined by

$$T(p(x)) = \begin{pmatrix} p(1) \\ \displaystyle\int_0^1 p(x)\,dx \end{pmatrix}.$$

(a) (2 points) Calculate $\mathcal{M}(T)$ for the standard bases.

*Solution.*Taking the transformation of each of the basis vectors in the domain, we have

$$\begin{aligned}
T(1) &= \begin{pmatrix} 1 \\ \displaystyle\int_0^1 1\,dx \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{e}_1 + \mathbf{e}_2 \\
T(x) &= \begin{pmatrix} x(1) \\ \displaystyle\int_0^1 x\,dx \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 \\
T(x^2) &= \begin{pmatrix} x^2(1) \\ \displaystyle\int_0^1 x^2\,dx \end{pmatrix} = \begin{pmatrix} 1 \\ 1/3 \end{pmatrix} = \mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 \\
\mathcal{M}(T) &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1/3 \end{pmatrix}
\end{aligned}$$

(b) (2 points) Is T one-to-one? Onto? Invertible?

Solution. To check if T is one-to-one, we note that $p(1) = 0$, so $p = a_1(x-1) + a_2(x-1)^2$. Then checking the remaining condition, we obtain

$$\int_0^1 a_1(x-1) + a_2(x-1)^2 dx = \left[\frac{a_1(x-1)^2}{2} + \frac{a_2(x-1)^3}{3}\right]_0^1 = \frac{a_1}{2} + \frac{a_2}{3} = 0.$$

Thus we have a one-dimensional null space, so T is not one-to-one, and hence is not invertible. From Theorem 2.4, we know that $\dim R(T) = \dim \mathcal{P}_2 - \dim N(T) = 3 - 1 = 2$. Since $\dim R(T) = 2 = \dim \mathcal{R}^2$, we have that T is onto.