MATH 672-010 Vector Spaces
Prof. D. A. Edwards Due: Sept. 29, 2008

# Homework Set 3 Solutions (Revised)

1. (4 points) Let $V$ be a vector space with dim $V = n$. Prove that there exist one-dimensional subspaces $U_i$ of $V$ such that

$$V = \bigoplus_{i=1}^{n} U_i.$$

*Solution.* Let $B = \{\mathbf{b}_i\}_1^n$ be a basis for $V$, and let $U_i = \text{Span } \mathbf{b}_i$. Since $U_i$ is spanned by one vector, $\dim U_i = 1$ and clearly $\dim V = \sum_{i=1}^{n} \dim U_i$. By the definition of a basis, any vector $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{b}_i.$$

But $c_i \mathbf{b}_i \in U_i$, so

$$V = \sum_{i=1}^{n} U_i.$$

We have now shown both hypotheses of Proposition 2.19, so by that proposition we have that

$$V = \bigoplus_{i=1}^{n} U_i.$$

2. (4 points) Let $V$ be a finite-dimensional subspace with subspaces $U_i$ such that

$$V = \bigoplus_{i=1}^{m} U_i.$$

Show that

$$\dim V = \sum_{i=1}^{m} \dim U_i.$$

*Solution.* We use induction on $m$, starting with $m = 2$. Since $V = U_1 \oplus U_2$, by proposition 1.9 $V = U_1 + U_2$ and $U_1 \cap U_2 = \mathbf{0}$. Thus by Theorem 2.18, $\dim V = \dim U_1 + \dim U_2 - \dim \mathbf{0} = \dim U_1 + \dim U_2$. Now assume true for $m$. Let $W = V \oplus U_{m+1}$, where $V$ is the previously proven case with dimension $m$. Then by proposition 1.9 $W = V + U_{m+1}$

and $V \cap U_{m+1} = 0$. Thus by Theorem 2.18, $\dim V = \dim V + \dim U_{m+1} - \dim \mathbf{0} = \dim V + \dim U_{m+1}$. But by assumption

$$\dim V = \sum_{i=1}^{m} \dim U_i,$$

so

$$\dim W = \sum_{i=1}^{m+1} \dim U_i,$$

as required.

3. Let $\mathbf{v} = (v_1, v_2, ..., v_n) \in \mathcal{R}^n$, $n > 1$. Let

$$W = \left\{ \mathbf{v} \in \mathcal{R}^n \middle| \sum_{i=1}^{n} v_i = 0 \right\}.$$

(a) (3 points) Verify that $W$ is a subspace of $\mathcal{R}^n$.

*Solution.* Let $\mathbf{v}$, $\mathbf{w} \in W$. Then checking the properties, we see that

$$\sum_{i=1}^{n} (\mathbf{v}+\mathbf{w})_i = \sum_{i=1}^{n} v_i + w_i = 0 + 0 = 0 \quad \Longrightarrow \quad \mathbf{v}+\mathbf{w} \in W, \tag{A1}$$

$$\sum_{i=1}^{n} (c\mathbf{v})_i = c\sum_{i=1}^{n} v_i = 0 \quad \Longrightarrow \quad c\mathbf{v} \in W, \tag{M1}$$

$$\sum_{i=1}^{n} (\mathbf{0})_i = 0 \quad \Longrightarrow \quad c\mathbf{v} \in W. \tag{A3}$$

Therefore $W$ is a subspace of $\mathcal{R}^n$.

(b) (5 points) Find a basis for $W$ and determine dim $W$.

*Solution.* Let $\mathbf{b}_i = \mathbf{e}_i - \mathbf{e}_{i+1}, i = 1, 2, \ldots n-1$.

$$\sum_{j=1}^{n} (\mathbf{b}_i)_j = 1 - 1 = 0,$$

so $\mathbf{b}_i \in W$. Checking to see if the set $B = \{\mathbf{b}_i\}_1^n$ is linearly independent, we have

$$\sum_{i=1}^{n-1} c_i \mathbf{b}_i = c_1 \mathbf{e}_1 + \sum_{i=2}^{n-1} (c_i - c_{i+1}) \mathbf{e}_i - c_n \mathbf{e}_n$$

Clearly $c_1 = 0$. Then checking the coefficient of $\mathbf{e}_2$, we see that $c_2 = 0$. Repeating this process, we see that we always obtain $c_i = 0$, so the set is linearly independent. Thus

$\dim W \geq n-1$ and Span $B \subseteq W$. But there exist $\mathbf{v} \in \mathcal{R}^n$ such that $\mathbf{v} \notin W$, so $W \neq \mathcal{R}^n$, $\dim W = n-1$, and $B$ is a basis for $W$.

4. Consider the following subspaces of $\mathcal{P}_3$:

$$V_1 = \{p(x) \in \mathcal{P}_3 | p(1) = 0\}.$$

$$V_2 = \{p(x) \in \mathcal{P}_3 | p(2) = 0\}.$$

(a) (2 points) Find a basis for $V_1$.

*Solution.* A basis for $V_1$ should contain polynomials of all degrees for which $p(1) = 0$ nontrivially. Thus we let $B_1 = \{(x-1)^i\}_1^3$. Clearly the set is linearly independent, since each basis vector is of a different degree than the previous ones. dim Span $B_1 = 3$, but $\dim V_1 < 4$ since there are constant polynomials not in $V_1$. Thus $V_1 =$ Span $B_1$.

(b) (2 points) Find a basis for $V_2$.

*Solution.* A basis for $V_2$ should contain polynomials of all degrees for which $p(2) = 0$ nontrivially. Thus we let $B_2 = \{(x-2)^i\}_1^3$. By the same arguments as in part (a), $B_2$ is a basis for $V_2$.

(c) (4 points) Find a basis for $V_1 \cap V_2$.

*Solution.* A basis for $V_1 \cap V_2$ should contain polynomials of all degrees for which $p(1) = 0$ and $p(2) = 0$ nontrivially. Thus we let $B_3 = \{(x-1)(x-2), (x-1)(x-2)^2\}$. These are linearly independent because they are of different degree. $V_1 \neq V_2$, so we know that $\dim(V_1 \cap V_2) < 3$. dim(Span $B_3$) $= 2$, so $B_3$ is a basis for $V_1 \cap V_2$.

(d) (3 points) Find a subspace $V_3$ such that $\mathcal{P}_3 = V_2 \oplus V_3$.

*Solution.* $V_2$ contains no constant polynomials, so let $V_3 = \mathcal{P}_0$. Then $V_2 \cap V_3 = \mathbf{0}$, and we just have to check if $\mathcal{P}_3 = V_2 + V_3$. We need to verify that any polynomial may be written in the following way:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = a_0 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3.$$

But this is equivalent to just letting $x = x - 1$ in the original polynomial. Since every polynomial can be shifted in this way, the result holds and $\mathcal{P}_3 = V_2 \oplus V_3$.

5. Consider the subspace of $\mathcal{P}_3$ spanned by

$$V = \left\{1 + 2x + 3x^2 - x^3, 2 - x + x^2 + 2x^3, 4 + 3x + 7x^2, 1 + 7x + 8x^2 - 5x^3\right\}.$$

(a) (2 points) Find a basis for Span $V$.

*Solution.* We simply need to reduce $V$ to a linearly independent set. Clearly the second polynomial is not a multiple of the first. Constructing the next test, we have

$$c_1(1 + 2x + 3x^2 - x^3) + c_2(2 - x + x^2 + 2x^3) = 4 + 3x + 7x^2 \qquad \Longrightarrow \qquad \begin{aligned} c_1 + 2c_2 &= 4 \\ 2c_1 - c_2 &= 3 \\ 3c_1 + c_2 &= 7 \\ -c_1 + 2c_2 &= 0 \end{aligned}$$

From the last equation, we have that $c_1 = 2c_2$. Substituting this result into the remaining equations yields $c_2 = 1$, $c_1 = 2$ as a consistent solution. Moreover, doing the next balance, we have that

$$c_1(1+2x+3x^2-x^3)+c_2(2-x+x^2+2x^3) = 1+7x+8x^2-5x^3 \quad \Longrightarrow \quad \begin{aligned} c_1 + 2c_2 &= 1 \\ 2c_1 - c_2 &= 7 \\ 3c_1 + c_2 &= 8 \\ -c_1 + 2c_2 &= -5 \end{aligned}$$

Solving the middle two equations together yields $c_1 = 3$. Substituting this result into the remaining equations yields $c_2 = 1$ as a consistent solution. Therefore, the latter two vectors are linearly dependent on the first two. Hence, a basis $B$ for $\operatorname{Span} V$ is given by the first two vectors, namely

$$B = \{1 + 2x + 3x^2 - x^3,\ 2 - x + x^2 + 2x^3\}.$$

(b) (4 points) Extend your answer to (a) to find a basis for $\mathcal{P}_3$.

*Solution.* We shall try $B_1 = \{1, x\}$. We simply need to confirm that $B \cup B_1$ is linearly independent, since $|B \cup B_1| = 4 = \dim \mathcal{P}_3$. Thus we solve the following equation:

$$c_1(1 + 2x + 3x^2 - x^3) + c_2(2 - x + x^2 + 2x^3) + c_3 + c_4x = 0.$$

Since the first two vectors are linearly independent, the only way to make the coefficients of $x_2$ and $x_3$ equal to zero is to take $c_1 = c_2 = 0$. This trivially forces $c_3 = c_4 = 0$, so $B \cup B_1$ is linearly independent.

6. Determine the conditions (if any) under which of the following are linear transformations:

   (a) (2 points) Let $A \in \mathcal{R}^{n\times n}$. Define $T : \mathcal{R}^{n\times n} \to \mathcal{R}^{n\times n}$ by $T(A) = AB - BA$ for a particular $B \in \mathcal{R}^{n\times n}$.

*Solution*: Checking the linear conditions, we have

$$\begin{aligned} T(A_1 + A_2) &= (A_1 + A_2)B - B(A_1 + A_2) = A_1B - BA_1 + A_2B - BA_2 \\ &= T(A_1) + T(A_2), \\ T(cA_1) &= (cA_1)B - B(cA_1) = cA_1B - cBA_1 = cT(A_1), \end{aligned}$$

and so the transformation is linear for any $B \in \mathcal{R}^{n\times n}$.

   (b) (2 points) Let $f(x) \in \mathcal{C}^n(\mathcal{R})$ (the vector space of $n$th continuously differentiable functions defined on $\mathcal{R}$). Define $T : \mathcal{C}^n(\mathcal{R}) \to \mathcal{P}_n$ by

$$T(f) = f(0) + \sum_{i=1}^{n} \frac{d^i f}{dx^i}(0)x^i.$$

*Solutions*: Checking the linear conditions, we have

$$\begin{aligned} T(f_1+f_2) &= [f_1(0)+f_2(0)] + \sum_{i=1}^{n} \frac{d^i(f_1+f_2)}{dx^i}(0)x^i \\ &= f_1(0) + \sum_{i=1}^{n} \frac{d^i f_1}{dx^i}(0)x^i + f_2(0) + \sum_{i=1}^{n} \frac{d^i f_2}{dx^i}(0)x^i = T(f_1)+T(f_2), \\ T(cf_1) &= (cf_1)(0) + \sum_{i=1}^{n} \frac{d^i(cf_1)}{dx^i}(0)x^i = c\left[f_1(0) + \sum_{i=1}^{n} \frac{d^i f_1}{dx^i}(0)x^i\right] = cT(f_1), \end{aligned}$$

and so the transformation is always linear.

(c) (3 points) Let $\mathbf{x} \in \mathcal{R}^n$. Define $T : \mathcal{R}^n \to \mathcal{R}^n$ by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for a particular $A \in \mathcal{R}^{n\times n}, \mathbf{b} \in \mathcal{R}^n$.

*Solution.* Checking the linear conditions, we have

$$\begin{aligned} T(\mathbf{x}_1+\mathbf{x}_2) &= A(\mathbf{x}_1+\mathbf{x}_2)+\mathbf{b} = A\mathbf{x}_1 + A\mathbf{x}_2 + \mathbf{b} = T(\mathbf{x}_1)+T(\mathbf{x}_2)-\mathbf{b} \\ T(c\mathbf{x}_1) &= A(c\mathbf{x}_1)+\mathbf{b} = cA\mathbf{x}_1+\mathbf{b} = cT(\mathbf{x}_1)-(c-1)\mathbf{b}, \end{aligned}$$

so the transformation is linear only if $\mathbf{b} = \mathbf{0}$.